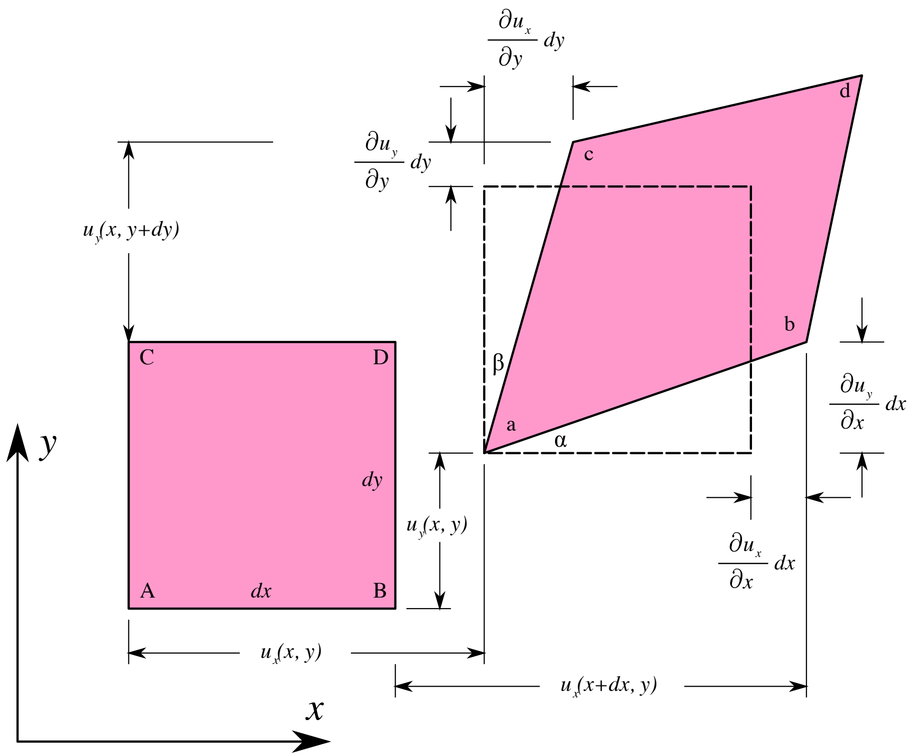
# Constitutive modeling

# Kinematics of strain




"2D geometric strain" by Sanpaz. Licensed under Public Domain via Wikimedia Commons.

# Strain tensor

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{for} \quad i = 1, 2, 3 \quad j = 1, 2, 3$$

# Volumetric strain

$$\varepsilon_{vol} = \mathrm{tr}(\boldsymbol{\varepsilon}) = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

# Material constants for isotropic materials

## Young's modulus

$$E = \frac{S_{11}}{\varepsilon_{11}}$$

## Bulk modulus

$$K = \frac{S_{11} + S_{22} + S_{33}}{3\varepsilon_{vol}}$$

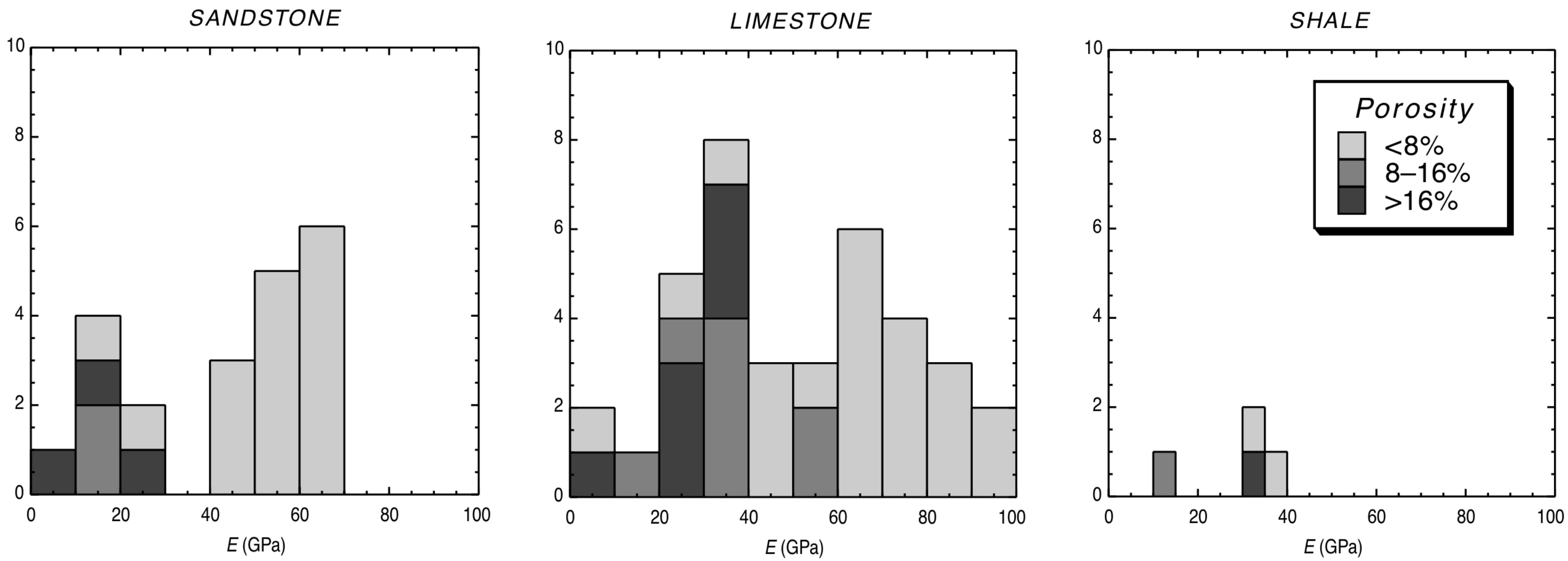
## Shear Modulus

$$G = \frac{1}{2}\frac{S_{13}}{\varepsilon_{13}}$$

## Poisson's ratio

$$\nu = \frac{\varepsilon_{33}}{\varepsilon_{11}}$$

# Typical Young's modulus values

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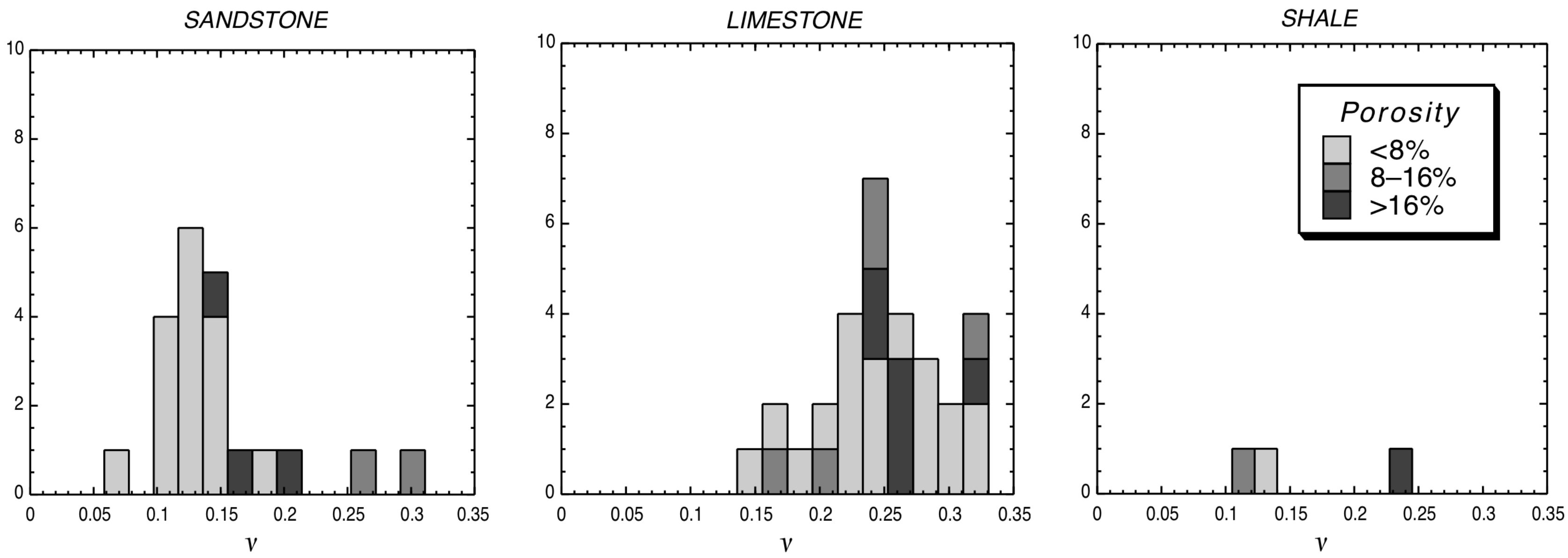
# Typical Poissons' modulus values



# Generalized Hooke's law

$$\vec{\sigma} = \boldsymbol{C}\vec{\varepsilon}$$

## For isotropic materials

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{Bmatrix}$$

$$\mathbf{S} = K\varepsilon_{vol}\mathbf{I} + 2G\left(\boldsymbol{\varepsilon} - \frac{1}{3}\varepsilon_{vol}\mathbf{I}\right)$$

$$G = \frac{E}{2(1+\nu)} \Rightarrow \text{shear modulus}$$

$$\mathbf{S} = \lambda\varepsilon_{vol}\mathbf{I} + 2G\boldsymbol{\varepsilon}$$

# Relationships between constants

| | $K =$ | $E =$ | $\lambda =$ | $G =$ | $\nu =$ | $M =$ |
|---|---|---|---|---|---|---|
| $(K,\ E)$ | $K$ | $E$ | $\frac{3K(3K-E)}{9K-E}$ | $\frac{3KE}{9K-E}$ | $\frac{3K-E}{6K}$ | $\frac{3K(3K+E)}{9K-E}$ |
| $(K,\ \lambda)$ | $K$ | $\frac{9K(K-\lambda)}{3K-\lambda}$ | $\lambda$ | $\frac{3(K-\lambda)}{2}$ | $\frac{\lambda}{3K-\lambda}$ | $3K - 2\lambda$ |
| $(K,\ G)$ | $K$ | $\frac{9KG}{3K+G}$ | $K - \frac{2G}{3}$ | $G$ | $\frac{3K-2G}{2(3K+G)}$ | $K + \frac{4G}{3}$ |
| $(K,\ \nu)$ | $K$ | $3K(1 - 2\nu)$ | $\frac{3K\nu}{1+\nu}$ | $\frac{3K(1-2\nu)}{2(1+\nu)}$ | $\nu$ | $\frac{3K(1-\nu)}{1+\nu}$ |
| $(K,\ M)$ | $K$ | $\frac{9K(M-K)}{3K+M}$ | $\frac{3K-M}{2}$ | $\frac{3(M-K)}{4}$ | $\frac{3K-M}{3K+M}$ | $M$ |
| $(E,\ \lambda)$ | $\frac{E+3\lambda+R}{6}$ | $E$ | $\lambda$ | $\frac{E-3\lambda+R}{4}$ | $\frac{2\lambda}{E+\lambda+R}$ | $\frac{E-\lambda+R}{2}$ |
| $(E,\ G)$ | $\frac{EG}{3(3G-E)}$ | $E$ | $\frac{G(E-2G)}{3G-E}$ | $G$ | $\frac{E}{2G} - 1$ | $\frac{G(4G-E)}{3G-E}$ |
| $(E,\ \nu)$ | $\frac{E}{3(1-2\nu)}$ | $E$ | $\frac{E\nu}{(1+\nu)(1-2\nu)}$ | $\frac{E}{2(1+\nu)}$ | $\nu$ | $\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$ |
| $(E,\ M)$ | $\frac{3M-E+S}{6}$ | $E$ | $\frac{M-E+S}{4}$ | $\frac{3M+E-S}{8}$ | $\frac{E-M+S}{4M}$ | $M$ |
| $(\lambda,\ G)$ | $\lambda + \frac{2G}{3}$ | $\frac{G(3\lambda+2G)}{\lambda+G}$ | $\lambda$ | $G$ | $\frac{\lambda}{2(\lambda+G)}$ | $\lambda + 2G$ |
| $(\lambda,\ \nu)$ | $\frac{\lambda(1+\nu)}{3\nu}$ | $\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$ | $\lambda$ | $\frac{\lambda(1-2\nu)}{2\nu}$ | $\nu$ | $\frac{\lambda(1-\nu)}{\nu}$ |
| $(\lambda,\ M)$ | $\frac{M+2\lambda}{3}$ | $\frac{(M-\lambda)(M+2\lambda)}{M+\lambda}$ | $\lambda$ | $\frac{M-\lambda}{2}$ | $\frac{\lambda}{M+\lambda}$ | $M$ |

# Siesmic wave velocity

$$V_p = \sqrt{\frac{M}{\rho}}, \qquad V_s = \sqrt{\frac{G}{\rho}}$$