# Linear Algebra Basics

# Matrix-Vector multiplication

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

$$c_1 = a_{11}b_1 + a_{12}b_2 + a_{13}b_3$$

$$c_2 = a_{21}b_1 + a_{22}b_2 + a_{23}b_3$$

$$c_3 = a_{31}b_1 + a_{32}b_2 + a_{33}b_3$$

**In words:** $c_i$ is the dot product of the $i^{\text{th}}$ row of **a** with $\vec{b}$...

# Matrix-Matrix multiplication

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$

$$c_{12} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}$$

$$\vdots$$

**In words:** $c_{ij}$ is the dot product of the $i^{\text{th}}$ row of **a** with the $j^{\text{th}}$ column of **b**

# Examples

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 7 \end{Bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 5 & 13 \end{bmatrix}$$

# The determinant of a 2 x 2 matrix

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(\mathbf{A}) = a \cdot d - b \cdot c$$

# The determinant of a 3 x 3 matrix

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\det(\mathbf{A}) = a \cdot (e \cdot i - f \cdot h) - b \cdot (d \cdot i - f \cdot g) + c \cdot (d \cdot h - e \cdot g)$$

# Matrix row operations

Used in solving matrix equations, i.e.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

$$\mathbf{A}\vec{x} = \vec{b}$$

- Swaping rows doesn't change solution
- Adding rows together doesn't change solution
- Multiplying row by a scalar doesn't change solution

# Example

Solve for $\vec{x}$

$$\begin{bmatrix} 2 & 3 & -2 \\ 0 & 0 & 3 \\ 1 & 0 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 6 \\ -6 \\ 3 \end{Bmatrix}$$

# Eigenvalue problem

$$\mathbf{A}\vec{v} = \lambda\vec{v}$$

$$\mathbf{A}\vec{v} - \lambda\vec{v} = 0$$

$$(\mathbf{A} - \lambda\mathbf{I})\vec{v} = 0$$

A non-trivial solution for $\vec{v}$ exists, if and only if

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

The $\lambda$'s are called the **eigenvalues**. Examples to follow in the context of stress.

# Vector transformation

$$\vec{v}' = \mathbf{Q}\vec{v}$$

$$\mathbf{Q} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

# Matrix transformation

$$\mathbf{S}' = \mathbf{Q}^{-1}\mathbf{S}\mathbf{Q}$$

If $\mathbf{Q}$ is chosen such that its columns are eigenvectors of $\mathbf{S}$, then $\mathbf{S}'$ will be *diagonal* with its entries cooresponding to the eigenvalues $\mathbf{S}$ (and $\mathbf{S}'$).