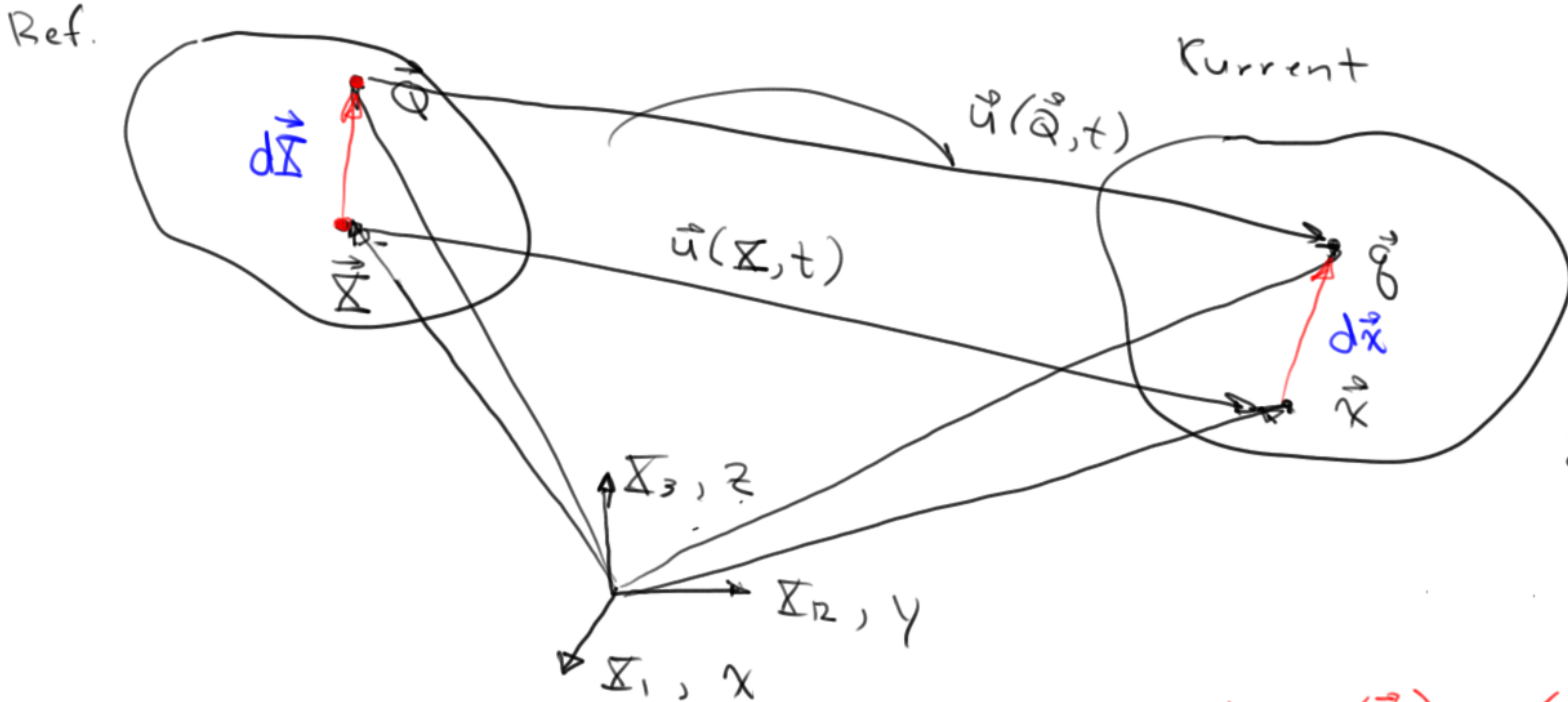


$$\delta_{ij} = \begin{cases} 1 & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$a_i = \delta_{ij} a_j$$
$$= a_i$$

$$\delta_{ij} Q_j - \delta_{ij} X_j =$$
$$Q_i - X_i$$

$$\vec{x} = \vec{X} + \vec{u}(\vec{X},t)$$
$$\vec{q} = \vec{Q} + \vec{u}(\vec{Q},t)$$

$$\vec{q} - \vec{x} = \vec{u}(\vec{Q}) - \vec{u}(\vec{X}) + (\vec{X} - \vec{Q})$$

Taylor expansion about $\vec{Q} = \vec{X}$

$$\underbrace{q_i - x_i}_{d\vec{x}} = Q_i - X_i + \frac{\partial u_i}{\partial X_j}(Q_j - X_j)$$
$$+ \frac{\partial^2 u_i}{\partial X_j \partial X_k}(Q_j - X_j)(Q_k - X_k)$$

$$\vec{x} = \vec{x}(\vec{X},t)$$
$$\vec{q} = \vec{q}(\vec{Q},t)$$

$$= \delta_{ij}(Q_j - X_j) + \frac{\partial u_i}{\partial X_j}\underbrace{(Q_j - X_j)}_{d\vec{X}} + \text{H.O.T.}$$

$$d\vec{x} = \left(\delta_{ij} + \frac{\partial u_i}{\partial X_j}\right)(Q_j - X_j) + O(\|\vec{Q} - \vec{X}\|)$$

$$\rightarrow d\vec{x}_i = \underbrace{\left(\delta_{ij} + \frac{\partial u_i}{\partial X_j}\right)} dX_j + \cancel{O(\|d\vec{X}\|^2)}^{\,0} \qquad \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

$F_{ij} \rightarrow$ Deformation gradient

$$F_{ij} = \delta_{ij} + \frac{\partial u_i}{\partial X_j}$$

$$d\vec{x} = F\,d\vec{X}$$

$$F = I + (\nabla \vec{u})^T = I + u\nabla_X$$

$$dx_1 = dX_1 + \frac{\partial u_1}{\partial X_1} dX_1 + \frac{\partial u_1}{\partial X_2} dX_2 + \frac{\partial u_1}{\partial X_3} dX_3$$

$$dx_2 = dX_2 + \frac{\partial u_2}{\partial X_1} dX_1 + \frac{\partial u_2}{\partial X_2} dX_2 + \frac{\partial u_2}{\partial X_3} dX_3$$

$$dx_3 = dX_3 + \frac{\partial u_3}{\partial X_1} dX_1 + \frac{\partial u_3}{\partial X_2} dX_2 + \frac{\partial u_3}{\partial X_3} dX_3$$

$$d\vec{x} = \begin{Bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{Bmatrix} = \begin{Bmatrix} dX_1 \\ dX_2 \\ dX_3 \end{Bmatrix} + \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3} \\ \frac{\partial u_2}{\partial X_1} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3} \\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & \frac{\partial u_3}{\partial X_3} \end{bmatrix} \begin{Bmatrix} dX_1 \\ dX_2 \\ dX_3 \end{Bmatrix}$$

$$\vec{u} = u_1 \hat{e}_1 + u_2 \hat{e}_2 + u_3 \hat{e}_3$$

$$\nabla_{\underline{X}} (\cdot) = \hat{e}_1 \frac{\partial(\cdot)}{\partial \underline{X}_1} + \hat{e}_2 \frac{\partial(\cdot)}{\partial \underline{X}_2} + \hat{e}_3 \frac{\partial(\cdot)}{\partial \underline{X}_3}$$

$$\begin{aligned}
\nabla_{\underline{X}} \vec{u} = \; & \hat{e}_1 \left[\frac{\partial(u_1 \hat{e}_1)}{\partial \underline{X}_1} + \frac{\partial(u_2 \hat{e}_2)}{\partial \underline{X}_1} + \frac{\partial(u_3 \hat{e}_3)}{\partial \underline{X}_1} \right] + \\
& \hat{e}_2 \left[\frac{\partial(u_1 \hat{e}_1)}{\partial \underline{X}_2} + \frac{\partial(u_2 \hat{e}_2)}{\partial \underline{X}_2} + \frac{\partial(u_3 \hat{e}_3)}{\partial \underline{X}_2} \right] + \\
& \hat{e}_3 \left[\frac{\partial(u_1 \hat{e}_1)}{\partial \underline{X}_3} + \frac{\partial(u_2 \hat{e}_2)}{\partial \underline{X}_3} + \frac{\partial(u_2 \hat{e}_2)}{\partial \underline{X}_3} \right]
\end{aligned}$$

Dyad (2^{nd}-order tensor)

$$A = \sum_i \sum_j a_i b_j \hat{e}_i \hat{e}_j$$

$$\nabla \vec{u} = \begin{bmatrix} \frac{\partial u_1}{\partial \underline{X}_1} & \frac{\partial u_2}{\partial \underline{X}_1} & \frac{\partial u_3}{\partial \underline{X}_1} \\ \frac{\partial u_1}{\partial \underline{X}_2} & \frac{\partial u_2}{\partial \underline{X}_2} & \frac{\partial u_3}{\partial \underline{X}_2} \\ \frac{\partial u_1}{\partial \underline{X}_3} & \frac{\partial u_2}{\partial \underline{X}_3} & \frac{\partial u_3}{\partial \underline{X}_3} \end{bmatrix}$$

$$d\vec{x} = F\,d\vec{X}$$

$$= \left(I + (\nabla \vec{u})^T\right) d\vec{X}$$

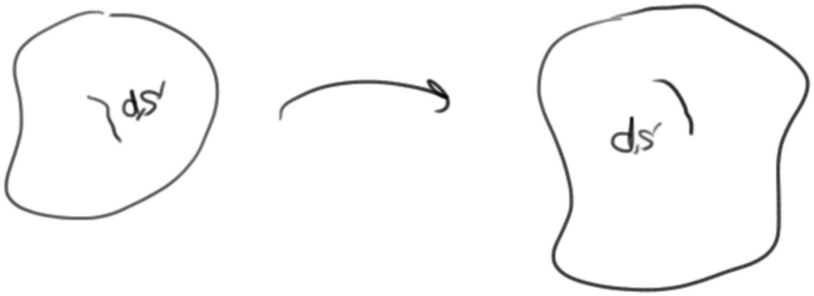
$$F^{-1} d\vec{x} = d\vec{X}$$

$$\vec{x} = \vec{X} + \vec{u}(\vec{X})$$

$$\frac{\partial \vec{x}_i}{\partial X_j} = \underbrace{\frac{\partial \vec{X}_i}{\partial X_j}} + \frac{\partial \vec{u}_i(\vec{X})}{\partial X_j}$$

$$\frac{\partial x_i}{\partial X_j} = \delta_{ij} + \frac{\partial u_i}{\partial X_j} = F_{ij}$$

$$F_{ij} = \frac{\partial x_i}{\partial X_j} = \frac{\partial x_i(X_1, X_2, X_3)}{\partial X_j}$$



$$(ds)^2 = |d\vec{x}|^2 = \left(\sqrt{dx_1^2 + dx_2^2 + dx_3^2}\right)^2 = \begin{bmatrix} dx_1 & dx_2 & dx_3 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = d\vec{x}^T d\vec{x}$$

$$(ds')^2 = d\vec{X}^T d\vec{X}$$

$$(ds)^2 - (ds')^2 = d\vec{x}^T d\vec{x} - d\vec{X}^T d\vec{X} = (F\,d\vec{X})^T (F\,d\vec{X}) - d\vec{X}^T d\vec{X}$$

$$= d\vec{X}^T F^T F\,d\vec{X} - d\vec{X}^T (I\,d\vec{X})$$

$$= d\vec{X}^T \boxed{(F^T F - I)}\, d\vec{X} \equiv 2E$$

$$E = \frac{1}{2}\left(F^T F - I\right) \rightarrow$$ Lagrangian Green-St. Venant