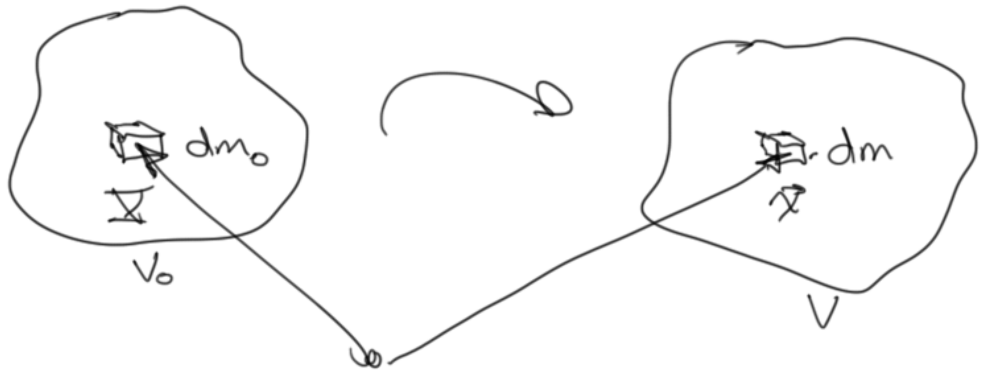
Mass conservation (Material Form)

$$\frac{D}{Dt}(\cdot) = \frac{\partial}{\partial t}(\cdot) + v_i \frac{\partial}{\partial x_i}(\cdot)$$
$$= \frac{\partial}{\partial t}(\cdot) + \vec{v} \cdot \nabla (\cdot)$$

Material time derivative



$$dm_0 = \rho_0 \, dV_0$$

$$\int_{V_0} \rho_0 \, dV_0 = \int \rho \, dV$$

$$dV_0 = dX_1 \, dX_2 \, dX_3$$
$$dV = dx_1 \, dx_2 \, dx_3$$

$$dV = |J| \, dV_0 = \left| \det\left(\frac{\partial \vec{x}_i}{\partial \vec{X}_j} \right) \right| dV_0$$
$$\Big| \underbrace{\det(F)}_{J} \Big| \, dV_0$$

$$\int_{V_0} \rho_0 \, dV_0 = \int \rho \, |J| \, dV_0$$

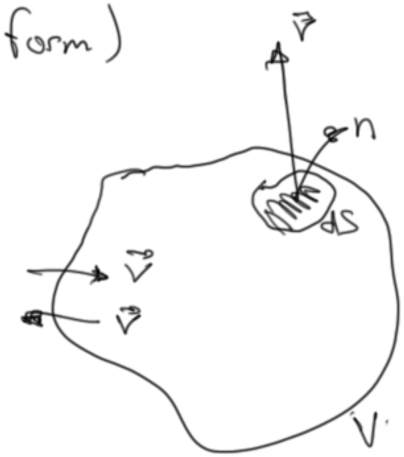
$$\rho_0 = \rho |J|$$
$$|J| = \frac{\rho_0}{\rho} \Rightarrow \boxed{J = \det(F) = \frac{\rho_0}{\rho}}$$

$$\det(F) = 1$$

Mass Conservation (differential form)

Divergence Theorem

$$\int_S \vec{v}\cdot\hat{n}\, dS = \int \nabla\cdot\vec{v}\, dV$$



$$\text{mass} = \int \rho\, dV = \int \rho(x_i, t)\, dV$$

time rate-of-change of mass $= \underbrace{\text{mass enters} - \text{mass exits}}_{\text{mass flux}}$

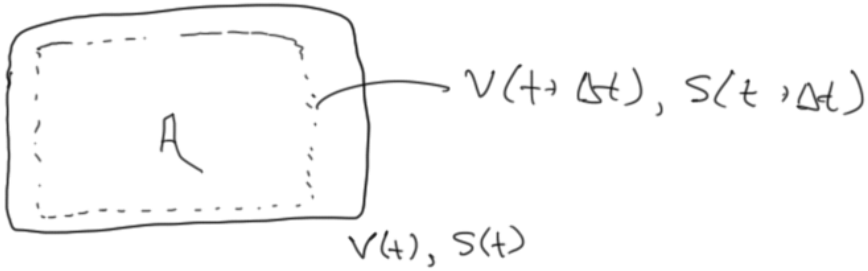
$$\frac{d}{dt}\int \rho\, dV = \underline{\int \frac{\partial \rho}{\partial t} dV} = -\int \rho\vec{v}\cdot\hat{n}\, dS = \underline{-\int \nabla\cdot(\rho\vec{v})\, dV}$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho\vec{v})}{\partial x_1} + \frac{\partial(\rho\vec{v})}{\partial x_2} + \frac{\partial(\rho\vec{v})}{\partial x_3} = 0 \Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i)$$

$$\underbrace{\frac{\partial \rho}{\partial t} + v_i\frac{\partial}{\partial x_i}\rho} + \rho\frac{\partial \vec{v}}{\partial x_i} = 0$$

$$\frac{D\rho}{Dt} + \rho\frac{\partial \vec{v}}{\partial x_i} = 0 \Rightarrow \boxed{\frac{D\rho}{Dt} + \rho\,\nabla\cdot\vec{v} = 0}$$



$$\frac{d}{dt}\int_{V(t)} \rho A \, dV$$

time rate-of-change of A = instantaneous change A at a constant + flux A

$$\frac{d}{dt}\int_{V(t)} \rho A \, dV = \underline{\int \frac{d}{dt}(\rho A)\, dV} + \underbrace{\int \rho A \vec{v} \vec{n} \, dS}_{\underline{\int \frac{\partial}{\partial x_i}(\rho A \vec{v})\, dV}}$$

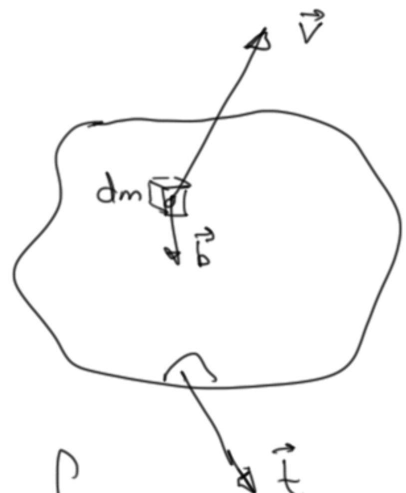
$$= \int A \frac{d}{dt}(\rho) + \rho \frac{d}{dt}(A) + A \, \nabla\cdot(\rho \vec{v}) + \rho \vec{v} \, \nabla A \, dV$$

$$= \int A \Big(\underbrace{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v})}_{=0}\Big) + \underbrace{\rho \frac{\partial A}{\partial t} + \rho \vec{v} \, \nabla A}_{\rho \frac{DA}{Dt}} \, dV$$

$$\boxed{\frac{d}{dt}\int_{V(t)} \rho A_i \, dV = \int \rho \frac{DA_i}{Dt} \, dV} \rightarrow \text{Reynolds Transport Theorem}$$

$$d\vec{p} = \vec{v}\, dm = \vec{v} \rho \, dV$$

$$p = \int \rho \vec{v} \, dV$$



body forces $= \int \rho \vec{b} \, dV$

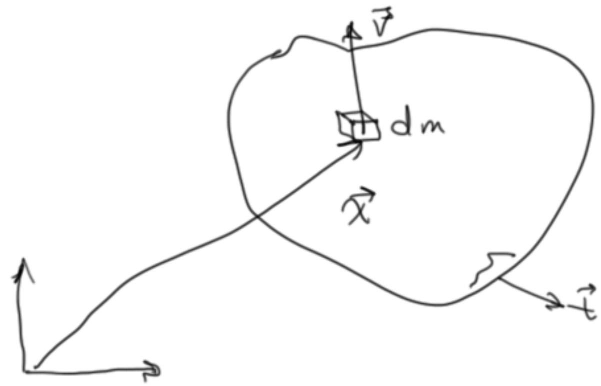
surface forces $= \int \vec{t} \, dS = \int \sigma^T \hat{n} \, dS$

$$\frac{d}{dt} p = \underbrace{\frac{d}{dt}\int \rho \vec{v} \, dV}_{\text{R.T.T.}} = \int \rho \vec{b} \, dV + \underbrace{\int \sigma^T \hat{n} \, dS}_{\text{D.T.}}$$

$$\int \rho \frac{D\vec{v}}{Dt} \, dV = \int \rho \vec{b} \, dV + \int \nabla \cdot \sigma^T \, dV = \int \left[\rho \vec{b} + \nabla \cdot \sigma^T \right] dV$$

$$\boxed{\rho \frac{D\vec{v}}{Dt} = \rho \vec{b} + \nabla \cdot \sigma^T}$$ Cauchy momentum eqn.

Angular momentum



$$\vec{x} \times \vec{v}\, dm$$

$$\int \vec{x} \times \rho \vec{v}\, dV = \int \vec{x} \times \rho \vec{b}\, dV + \int \vec{x} \times \vec{t}\, dS$$

$$\vec{a} \cdot \vec{b} = a_i b_i$$

$$\vec{a} \times \vec{b} = \varepsilon_{ijk} a_j b_k$$

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$$\varepsilon_{ijk} = \begin{cases} 1 & \text{even permutation} \\ & 132,\ 213 \\ -1 & \text{odd permutation} \\ & 113 \\ 0 & \text{repeated} \end{cases}$$

$$\frac{d}{dt}\int \varepsilon_{rmn} x_m v_n \rho\, dV = \int \varepsilon_{rmn} x_m t_n\, dS + \int \varepsilon_{rmn} x_m \rho b_m\, dV$$

$$\int \varepsilon_{rmn} \rho \frac{D(x_m v_n)}{Dt} dV = \int \cancelto{0}{\rho\, \varepsilon_{rmn} v_n \underbrace{\frac{Dx_m}{Dt}}_{v_m}} + \underbrace{\rho\, \varepsilon_{rmn} x_m \frac{Dv_n}{Dt}\, dV}_{= \text{L.H.S.}}$$

$$\int \varepsilon_{rmn}\, x_m t_n \, dS = \int \varepsilon_{rmn}\, x_m\, \sigma_{jn}\, n_j \, dS = \int \varepsilon_{rmn} \frac{\partial}{\partial x_j}(x_m \sigma_{jn})\, dV$$

$$t_n = \sigma_{jn} \hat{n}_j$$

$$= \int \varepsilon_{rmn}\, \sigma_{jn} \frac{\partial}{\partial x_j} x_m + \varepsilon_{rmn}\, x_m \frac{\partial \sigma_{jn}}{\partial x_j}\, dV \qquad 1^{st} \text{ term R.H.S.}$$

$$\int \rho\, \varepsilon_{rmn}\, x_m \frac{DV_n}{Dt}\, dV = \int \varepsilon_{rmn}\, \sigma_{jn} \frac{\partial x_m}{\partial x_j} + \varepsilon_{rmn}\, x_m \frac{\partial \sigma_{ji}}{\partial x_j}\, dV + \int \varepsilon_{rmn}\, x_m\, \rho b_n \, dV$$

$$\int \varepsilon_{rmn}\, x_m \underbrace{\left[\rho \frac{DV_n}{Dt} - \frac{\partial \sigma_{jn}}{\partial x_j} - \rho b_n\right]}_{=0} dV = \int \varepsilon_{rmn}\, \sigma_{jn} \frac{\partial x_m}{\partial x_j}\, dV$$

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 \end{cases}$$

$$0 = \int \varepsilon_{rmn}\, \sigma_{jn}\, \delta_{mj}\, dV = \varepsilon_{rmn}\, \sigma_{mn}$$

$$r=1 \rightarrow 0 = \sigma_{23} - \sigma_{32}$$

$$r=2 \rightarrow 0 = \sigma_{31} - \sigma_{13}$$

$$r=3 \rightarrow 0 = \sigma_{21} - \sigma_{21}$$

$$\boxed{\sigma^T = \sigma \quad \text{or} \quad \sigma_{jm} = \sigma_{mj}} \quad \sigma \text{ is symm.}$$

Energy

time rate-of-change of energy = mechanical work + heat + radiation

K.E. : $\frac{1}{2} dm\, \vec{v} \cdot \vec{v} = \frac{1}{2} \rho \vec{v} \cdot \vec{v}\, dV$

Internal energy : $\rho u\, dV$

$$\frac{d}{dt} \int \left(\frac{1}{2} \rho \vec{v} \cdot \vec{v} + \rho u \right) dV = \int \rho \frac{D(\vec{v} \cdot \vec{v})}{Dt} dV + \int \rho \frac{Du}{Dt}\, dV = \text{L.H.S.}$$

mechanical work (power) – body forces $\rightarrow \int \rho \vec{b} \cdot \vec{v}\, dV$

" " surfaces forces $\rightarrow \int \vec{t} \cdot \vec{v}\, dS = \int \sigma_{ji} n_j v_i\, dS$

$$= \int \frac{\partial}{\partial x_j} (\sigma_{ji} v_i)\, dV$$

$$= \int \frac{\partial \sigma_{ji}}{\partial x_j} v_i + \sigma_{ji} \underbrace{\frac{\partial v_i}{\partial x_j}}_{L_{ij}} dV$$

heat cond : $-\int \vec{q} \cdot \vec{n}\, dS = -\int \nabla \cdot \vec{q}\, dV$

radiation : $\int \rho r\, dV$

$$\int \rho\left(\vec{v}\,\frac{D\vec{v}}{Dt} + \frac{Du}{Dt}\right) dV = \int \left[\rho \vec{b}\,\vec{v} + \nabla\cdot\sigma^T \vec{v} + \sigma^T : L - \nabla\vec{q} + \rho r\right] dV$$

$$\int \vec{v}\,\underbrace{\left[\rho\frac{D\vec{v}}{Dt} - \nabla\sigma^T - \rho\vec{b}\right]}_{=0} + \rho\frac{Du}{Dt}\, dV = \int \sigma^T : L - \nabla\cdot\vec{q} + \rho r\, dV$$

$$L = \overset{\text{symm}}{D} + \overset{\text{antisymm}}{W}$$

$$\sigma : W = 0$$

$$\rho\frac{Du}{Dt} = \sigma^T : L - \nabla\cdot\vec{q} + \rho r$$

$$\boxed{\rho\frac{Du}{Dt} = \sigma : D - \nabla\cdot\vec{q} + \rho r}$$ Energy

$$\boxed{\rho\frac{D\vec{v}}{Dt} = \nabla\cdot\sigma + \rho\vec{b}}$$ Linear momentum

$$\boxed{\sigma^T = \sigma}$$ Angular Momentum

$$\boxed{\det(F) = \frac{\rho_0}{\rho} \quad \text{or} \quad \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0 \quad \text{or} \quad \frac{D\rho}{Dt} + \rho\,\nabla\cdot\vec{v} = 0}$$ Mass