2nd Law

rate of entropy increase $\geq$ entropy input rate

specific entropy (entropy/volume)

$$\frac{d}{dt}\underbrace{\int s\rho \, dV}_{S} \geq \int \frac{\rho r}{T} dV + \int -\frac{\vec{q}}{T}\cdot\hat{n}\, dS$$

$$\int \rho \frac{Ds}{Dt} dV \geq \int \frac{\rho r}{T} dV + \int -\nabla\cdot\left(\frac{\vec{q}}{T}\right) dV$$

$$\boxed{\frac{Ds}{Dt} \geq \frac{r}{T} - \frac{1}{\rho}\nabla\cdot\left(\frac{\vec{q}}{T}\right)}$$

Clausis-Duhem Inequality

Recall Conservation of Momentum

$$\rho \frac{D\vec{v}}{Dt} = \nabla \cdot \sigma + \rho \vec{b} \qquad \sigma^T = \sigma$$

$$\frac{D\vec{u}}{Dt} = \vec{v}$$

$$\left.\begin{aligned}
\rho \frac{D^2 u_1}{Dt^2} &= \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + \rho b_1 \\
\rho \frac{D^2 u_2}{Dt^2} &= \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + \rho b_2 \\
\rho \frac{D^2 u_3}{Dt^2} &= \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho b_3
\end{aligned}\right\} \quad 3 \text{ eqns.} \qquad 9 \text{ unknowns}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \quad , \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_1} \quad , \quad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

$$\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

$$\varepsilon_{23} = \varepsilon_{32} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$$

$$\varepsilon_{13} = \varepsilon_{31} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)$$

Constitutive model

$$\sigma \sim \varepsilon$$

Assume thermodynamic equation-of-state

$$\rightarrow u = u(s, \vec{\Omega}, \vec{\chi})$$
$$= u(s, \vec{\Omega})$$

Define

$$T \equiv \left(\frac{\partial u}{\partial s}\right)_{\Omega} \quad , \quad \tau_j = \left(\frac{\partial u}{\partial \Omega_j}\right)_s$$

For a change in u

$$du = \left(\frac{\partial u}{\partial s}\right) ds + \left(\frac{\partial u}{\partial \Omega_j}\right) d\Omega_j$$

$$= \underbrace{T\,ds + \tau_j\, d\Omega_j}_{\text{Gibbs relation}}$$

$\Omega = v$ - specific volume, s

$du = T\,ds - p\,dv$

$-p = \left(\frac{\partial u}{\partial v}\right)_s$

For a given $\underline{X}$

$$T = T(s, \vec{\Lambda}) \qquad \boxed{\tau_j = \tau_j(s, \vec{\Lambda})}$$

assume that these relationships are invertable, i.e.

$$s = s(T, \vec{\Lambda})$$

$$u = u(s, \vec{\Lambda}) \qquad \tau_j = \tau_j(T, \vec{\Lambda})$$

$$= u(T, \vec{\Lambda}) \qquad \Lambda_j = \Lambda_j(T, \vec{\tau})$$

Thermodynamic potential Helmholtz free energy

$$\phi = u - sT$$

this is amount of obtained/stored energy in a closed system at constant T. We chose T, Λ_j to be independent quantities...

We can show

$$d\psi = -s\,dT + \tau_j\,d\lambda_j$$

$$\dot{\psi} = -s\cancel{\dot{T}}^{\,0} + \tau_j\,\dot{\lambda}_j$$

$$\dot{\psi} = \tau_j\,\dot{\lambda}_j \qquad (*)$$

$$\rho\dot{\psi} = \sigma_{ij}\,\dot{\varepsilon}_{ij}$$
$$= \tau_j\,\dot{\lambda}_j$$

Energy Eqn.

$$\rho\frac{Du}{Dt} = \sigma_{ij}D_{ij} - \frac{\partial q_j}{\partial x_j} + \rho r$$

$$\psi = u - sT \quad\Rightarrow\quad \boxed{u = \psi + sT}$$

$(**)$

$$\rho\dot{\psi} = \sigma_{ij}D_{ij} + \rho r - \frac{\partial q_j}{\partial x_j} - \rho\dot{s} - \cancel{\rho\dot{T}}^{\,0} = \sigma_{ij}D_{ij} \simeq \sigma_{ij}\dot{\varepsilon}_{ij}$$

C-D

$$\dot{s} = \frac{1}{T}\left[r - \frac{1}{\rho}\frac{\partial q_j}{\partial x_j}\right]$$

$$\frac{\partial}{\partial \dot{\varepsilon}_{ij}} \rho \dot{\psi} = \frac{\partial}{\partial \varepsilon_{ij}} \sigma_{ij} \dot{\varepsilon}_{ij}$$

$$\rho \frac{\partial \dot{\psi}}{\partial \dot{\varepsilon}_{ij}} = \sigma_{ij} \cancel{\frac{\partial \dot{\varepsilon}_{ij}}{\partial \dot{\varepsilon}_{ij}}} \; \delta_{ij}$$

$$\rho \frac{\partial \psi}{\partial \varepsilon_{ij}} = \sigma_{ij}$$

$W = \rho \psi \Rightarrow$ strain energy density function

$$\sigma_{ij} = \frac{\partial W(\varepsilon_{kl})}{\partial \varepsilon_{ij}}$$

$$\dot{\sigma}_{ij} = \underbrace{\frac{\partial W}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}}}_{C_{ijkl}} \dot{\varepsilon}_{kl}$$

$C_{ijkl} \rightarrow 4^{th}$ order tensor, $3 \times 3 \times 3 \times 3 \Rightarrow 81$ components

$\boxed{\sigma_{ij} = C_{ijkl} \varepsilon_{kl}} \longrightarrow$ Generalized Hooke's Law

$$\sigma_{ij} = \sigma_{ji}$$

$$C_{ijkl} = C_{jikl} \rightarrow 81 \text{ to } 54$$

$$\varepsilon_{ij} = \varepsilon_{ji}$$

$$C_{ijkl} = C_{ijlk} \rightarrow 54 \text{ to } 36$$

$$C_{ijkl} = \frac{\partial w}{\partial \varepsilon_{ij} \, \partial \varepsilon_{kl}}$$

$$C_{ijkl} = C_{klij} \rightarrow 36 \text{ to } 21$$