$$S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

$$\sigma_{ij} = +\frac{1}{3}\sigma_{kk} + S_{ij}$$

$$\boxed{\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl}} \rightarrow \text{Generalized Hooke's law}$$

$$\vec{\sigma} = C\vec{\varepsilon}$$

$$C_{ijkl} = C_{ijlk} \qquad \frac{\partial W}{\partial \varepsilon_{ij}\partial \varepsilon_{kl}} = C$$

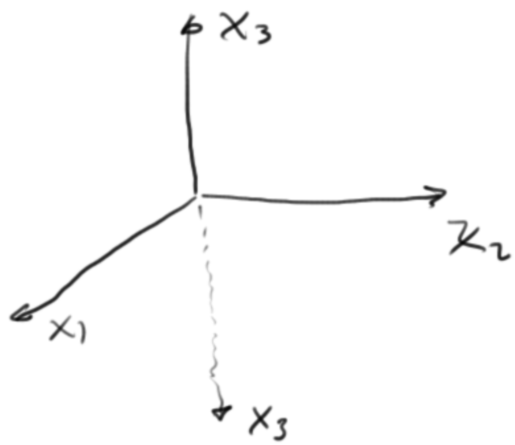
$$C_{ijkl} = C_{jikl}$$

$$C_{ijkl} = C_{klij} \rightarrow 21 \text{ components}$$

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} \\ & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\ & & C_{3333} & C_{3323} & C_{3331} & C_{3312} \\ & \text{symm.} & & C_{2323} & C_{2331} & C_{2312} \\ & & & & C_{3131} & C_{3112} \\ & & & & & C_{1212} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{Bmatrix}$$

triclinic

Consider a plane of symm.



$x_1 - x_2$ plane is a plane of symm.

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\sigma' = R \sigma R^T = \begin{bmatrix} \sigma_{11} & \sigma_{12} & -\sigma_{31} \\ & \sigma_{22} & -\sigma_{23} \\ & & \sigma_{33} \end{bmatrix}$$

similarly for $\varepsilon' = R \varepsilon R^T$

$\boxed{\varepsilon'_{31} = -\varepsilon_{31}}$ + $\boxed{\varepsilon'_{23} = -\varepsilon_{23}}$

$$\sigma'_{11} = C_{1111}\varepsilon'_{11} + C_{1122}\varepsilon'_{22} + C_{1133}\varepsilon'_{33} + 2C_{1123}\varepsilon'_{23} + 2C_{1131}\varepsilon'_{31} + 2C_{1121}\varepsilon'_{12}$$

$$\sigma'_{11} = C_{1111}\varepsilon_{11} + C_{1122}\varepsilon_{22} + C_{1133}\varepsilon_{33} - 2C_{1123}\varepsilon_{23} - 2C_{1131}\varepsilon_{31} + 2C_{1121}\varepsilon_{12}$$

$$\sigma_{11} = C_{1111}\varepsilon_{11} + C_{1122}\varepsilon_{22} + C_{1133}C_{33} + 2C_{1123}\varepsilon_{23} + 2C_{1131}\varepsilon_{33} + 2C_{1121}\varepsilon_{12}$$

$=$

$$0 = -4\, C_{1123}\, \varepsilon_{23} - 4\, C_{1131}\, \varepsilon_{31}$$

$$C_{1123} = C_{1131} = 0$$

Similar argument

$$C_{2222} = C_{2231} = C_{3323} = C_{3331} = 0$$

$$C = \begin{bmatrix} C_{1111} & C_{1112} & C_{1133} & 0 & 0 & C_{1112} \\ & C_{2232} & C_{2233} & 0 & 0 & C_{2212} \\ & & C_{3333} & 0 & 0 & C_{3312} \\ & & & C_{2323} & C_{2331} & 0 \\ & \text{symm.} & & & C_{3131} & 0 \\ & & & & & C_{1212} \end{bmatrix}$$

monoclinic material $\Rightarrow$ 13 independent constants

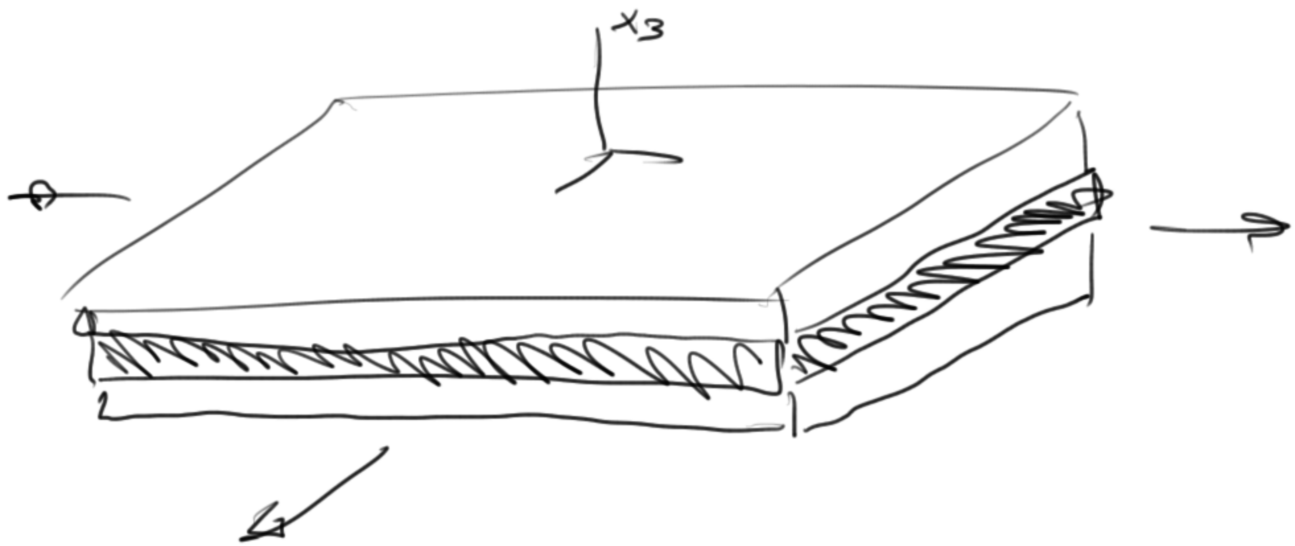
If 3 orthogonal planes of symm.

$$C_{1122} = C_{2223} = C_{2231} = C_{2212} = 0$$

9 independent constants → orthotropic material

If there exists an axes about which a material has identical properties then $\underline{\underline{5}}$ independent constants

transversely isotropic



For a material in which every plane is a plane of symm.

Isotropic material

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{23} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ & & \lambda+2\mu & 0 & 0 & 0 \\ & \text{symm.} & & \mu & 0 & 0 \\ & & & & \mu & 0 \\ & & & & & \mu \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{Bmatrix}$$

$$\lambda = \frac{2\mu}{1-2\mu} = \frac{\mu(E-2\mu)}{3\mu - E} = K - \frac{2}{3}\mu$$

$\lambda \rightarrow$ Lamé's constant

$K \rightarrow$ Bulk Modulus

$E \rightarrow$ Young's Modulus

$\mu\ (G) \Rightarrow$ shear modulus

$\nu \rightarrow$ Poisson's ratio.

For isotropic materials

$$C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk})$$

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} = [\lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ij}\delta_{kl} + \underbrace{\delta_{il}\delta_{jk})]\varepsilon_{kl}}$$

$$\delta_{il}\,\underbrace{\delta_{jk}\,\varepsilon_{kl}}$$

$$\delta_{il}\,\varepsilon_{jl}$$

$$\varepsilon_{ji}$$

$$\boxed{\sigma_{ij} = \lambda \delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}} \leftarrow \varepsilon_{ij}$$

let $j = i$

$$\sigma_{ii} = \lambda \underbrace{\delta_{ii}}_{3}\varepsilon_{kk} + 2\mu\varepsilon_{kk}$$

$$\sigma_{kk} = (3\lambda + 2\mu)\varepsilon_{kk} \Rightarrow \boxed{\varepsilon_{kk} = \frac{\sigma_{kk}}{(3\lambda + 2\mu)}}$$

$$\boxed{\varepsilon_{ij} = \frac{-\nu}{E}\sigma_{kk}\delta_{ij} + \frac{1+\nu}{E}\sigma_{ij}}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$$

$$\mu = \frac{E}{2(1+\nu)}$$

$$\varepsilon_{11} = \frac{1}{E}\left[\sigma_{11} - \nu(\sigma_{22} + \sigma_{33})\right]$$

$$\varepsilon_{22} = \frac{1}{E}\left[\sigma_{22} - \nu(\sigma_{11} + \sigma_{33})\right]$$

$$\varepsilon_{33} = \frac{1}{E}\left[\sigma_{33} - \nu(\sigma_{11} + \sigma_{22})\right]$$

$$\varepsilon_{23} = \frac{1}{2\mu}\sigma_{23}$$

$$\varepsilon_{31} = \frac{1}{2\mu}\sigma_{31}$$

$$\varepsilon_{12} = \frac{1}{2\mu}\sigma_{12}$$

$$\vec{F} = \sigma^T \vec{n}$$

<u>Plane stress</u> $(\sigma_{33} = \sigma_{13} = \sigma_{23} = 0)$

$$\underbrace{\begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{12}\end{Bmatrix}}_{\vec{\varepsilon}} = \frac{1}{E}\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & (1+\nu)\end{bmatrix}\underbrace{\begin{Bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{23}\end{Bmatrix}}_{\vec{\sigma}} \Rightarrow \left\{\vec{\sigma}\right\} = \begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & 1-\nu\end{bmatrix}\left\{\vec{\varepsilon}\right\}$$