Mass balance fluid

$$\frac{D\rho^f}{Dt} + \rho^f \nabla_x \cdot \vec{v}^f = 0$$

$$\frac{D(\phi^f \rho_f)}{Dt} + (\phi^f \rho_f) \nabla_x \cdot \vec{v}^f = 0$$

$$\underbrace{\frac{D(\phi^f \rho_f)}{Dt} + (\phi^f \rho_f) \nabla_x \cdot \vec{v}^s} + \nabla_x \cdot \vec{\omega}$$

$$\frac{D(\cdot)}{Dt} = \frac{\partial(\cdot)}{\partial t} + (\cdot) \cdot \vec{v}$$

$$\frac{D^s(\phi^f \rho_f)}{Dt} + \nabla_x \cdot \vec{\omega} = 0$$

$$\frac{D}{Dt}\int \phi^f \rho_f \, dV + \int \vec{\omega} \cdot n \, dS = 0$$

$$\frac{D}{Dt}(\Phi^f \rho_f) = -\nabla_X \cdot \vec{W} \qquad (*)$$

$$\rho^f = \frac{\text{mass fluid}}{\text{total volume}} \qquad \rho_f = \frac{\text{mass fluid}}{\text{volume fluid}}$$

$$\rho^f = \phi^f \rho_f$$

relative mass flux in/out of the solid skeleton

$$\vec{\omega} = \phi^f \rho_f (\vec{v}^f - \vec{v}^s)$$

$$\vec{v}^f = \frac{1}{\phi^f \rho_f} \vec{\omega} + \vec{v}^s$$

Nanson's relation → Piola transformation

$$\vec{W} = J F^{-1} \cdot \vec{\omega}$$

Lagrangian porosity

$$\phi^f = \frac{\text{current fluid volume}}{\text{current total volume}}$$

$$\Phi^f = \frac{\text{current fluid volume}}{\text{reference total volume}}$$

$$\boxed{\Phi^f = J\phi^f} \leftarrow \text{Lagrangian}$$

LHS of $\star$

$$\frac{D(\Phi^f \rho^f)}{Dt} = \dot{\Phi}^f \rho_f + \dot{\rho}_f \Phi^f$$

Assume barotropic

$$\rho_f = \rho_f(p)$$

$$\dot{\rho}_f = \frac{\partial \rho_f}{\partial p^f} \frac{Dp^f}{Dt}$$

$$= \left[\frac{\rho_f}{\cancel{K_f}}\right] \cancel{K_f} \frac{D}{Dt} \log J$$

$$= \rho_f \frac{D}{Dt} \log J$$

Recall

$$\frac{1}{K_f} = c_f = \frac{1}{\rho_f}\frac{\partial \rho_f}{\partial p^f} \Rightarrow \frac{\partial \rho_f}{\partial p^f} = \frac{\rho_f}{K_f}$$

$$\int_{\rho_{f_0}}^{\rho_f} \frac{K_f}{\rho_f}\, d\rho_f = \int_{p_0^f}^{p^f} dp$$

$$J = \det(F) = \frac{\rho_f}{\rho_{f_0}} = \exp\left(\frac{p^f - p_0^f}{K_f}\right)$$

$$\frac{D}{Dt}\left(K_f \log J\right) = \frac{D}{Dt}\left(p^f - p_0^f\right)$$

$$\frac{D}{Dt}(p^f) = K_f \frac{D}{Dt} \log J$$

$$\frac{D}{Dt}(\Phi^f) = \frac{D}{Dt}\left[\alpha \log J + \frac{\alpha - \Phi^f}{K_s} p^f\right]$$

Athy's pressure-porosity relation assuming small change in porosity over infintisimal time

$$\Phi = \Phi_0 e^{(k \ldots)}$$

$$\frac{D\Phi^f}{Dt} = \frac{K_s}{K_s + p^f}\left(\frac{D\alpha}{Dt}\left(\log J + \frac{p^f}{K_s}\right)\right) + \frac{K_s}{K_s + p^f}\left(\frac{\alpha}{J}\frac{DJ}{Dt} + \frac{\alpha - \Phi^f}{K_s}\frac{Dp^f}{Dt}\right)$$

Assume $p^f \ll K_s$

$$\frac{D\Phi^f}{Dt} = \frac{D\alpha}{Dt}\left(\log J + \frac{p^f}{K_s}\right) + \frac{\alpha}{J}\frac{DJ}{Dt} + \frac{\alpha - \Phi^f}{K_s}\frac{Dp^f}{Dt}$$

$$\rho_f\left(\frac{D\alpha}{Dt}\left(\log J + \frac{p^f}{K_s}\right) + \frac{\alpha}{J}\frac{DJ}{Dt} + \frac{1}{M}\frac{Dp^f}{Dt}\right) = -\nabla_X \cdot \vec{W}$$

where

$$M = \frac{K_s K_f}{K_f(\alpha - \Phi^f) + K_s \Phi^f}$$ is Biot's Modulus

Use Darcy's law

$$\vec{v}^f - \vec{v}^s = \frac{1}{\phi^f \rho_f}\vec{w} = \frac{\bar{\bar{k}}}{\mu}\cdot\left[-\nabla_x \cdot p^f + \phi^f \rho_f \vec{g}\right]$$ ← Eulerian

$$\frac{1}{\Phi^f \rho_f}\vec{W} = \frac{\bar{\bar{K}}}{\mu}\cdot\left[-\nabla_X \cdot p^f + \Phi^f \rho_f F^T \vec{g}\right]$$ ← Lagrangian

$$\bar{\bar{K}} = J\, F^{-1} \cdot \bar{\bar{k}} \cdot F^{-T}$$ $\bar{\bar{k}}$ → permeability tensor