Assume $p^f \ll K_s$

$$\frac{D\Phi^f}{Dt} = \frac{D\alpha}{Dt}\left(\log J + \frac{p^f}{K_s}\right) + \frac{\alpha}{J}\frac{DJ}{Dt} + \frac{\alpha - \Phi^f}{K_s}\frac{Dp^f}{Dt}$$

$$\rho^f\left(\frac{D\alpha}{Dt}\left(\log J + \frac{p^f}{K_s}\right) + \frac{\alpha}{J}\frac{DJ}{Dt} + \frac{1}{M}\frac{Dp^f}{Dt}\right) = -\nabla_X \cdot \vec{W}$$

where

$$M = \frac{K_s K_f}{K_f(\alpha - \Phi^f) + K_s \Phi^f}$$ is Biot's Modulus

Use Darcy's law

$$\vec{v}^f - \vec{v}^s = \frac{1}{\cancel{\phi^f}\rho_f}\vec{w} = \frac{\bar{\bar{k}}}{\mu}\cdot\left[-\nabla_x \cdot p^f + \phi^f \rho_f \vec{g}\right]$$ Eulerian

$$\frac{1}{\cancel{\Phi^f}\rho_f}\vec{W} = \frac{\bar{\bar{K}}}{\mu}\cdot\left[-\nabla_X \cdot p^f + \Phi^f \rho_f F^T \vec{g}\right]$$ Lagrangian

$$\bar{\bar{K}} = J F^{-1} \cdot \bar{\bar{k}} \cdot F^{-T}$$ $\bar{\bar{k}}$ = permeability tensor

$$\rho_f \left(\underbrace{\frac{D\alpha}{Dt}\left(\log J + \frac{p_f}{K_s}\right)}_{=0} + \underbrace{\frac{\alpha}{J}\frac{DJ}{Dt}} + \frac{1}{M}\frac{Dp^f}{Dt} \right) = -\nabla_X \vec{W}$$

$$\frac{D\alpha}{Dt} \qquad \alpha = 1 - \frac{K}{K_s}$$

$$J = \frac{\rho_0}{\rho}$$

$$\frac{\alpha}{J}\frac{DJ}{\cancel{Dt}} = \frac{\alpha}{\cancel{J}}\dot{\varepsilon}_{ii}\cancel{J} = \alpha\,\dot{\varepsilon}_{ii}$$

$$\frac{D}{Dt}(\rho J) = \cancel{\frac{D}{Dt}(\rho_0)} = 0$$

$$J\frac{D\rho}{Dt} + \rho\frac{DJ}{Dt} = 0$$

$$\rho_f\left[\alpha\,\dot{\varepsilon}_{ii} + \frac{1}{M}\dot{p}_f\right] + \nabla\vec{\omega} = 0$$

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{v} = 0$$

$$\frac{1}{\rho_f}\vec{\omega} = \frac{\bar{\bar{K}}}{\mu}\cdot\left[-\nabla p^f\right] \quad \text{ignoring gravity}$$

$$J\left(-\cancel{\rho}\nabla\cdot\vec{v}\right) + \cancel{\rho}\frac{DJ}{Dt} = 0$$

$$\boxed{\alpha\,\dot{\varepsilon}_{ii} + \frac{1}{M}\dot{p}_f + \nabla\cdot\left(-\frac{\bar{\bar{K}}}{\mu}\nabla p^f\right) = 0 \qquad \nabla\cdot\sigma + \rho\vec{b} = 0}$$

$$\boxed{\frac{d\rho}{dt} + \nabla\cdot(\rho\vec{v}) = 0 \qquad \frac{D\rho}{Dt} + \rho\nabla\cdot\vec{v} = 0}$$

$$\frac{DJ}{Dt} = J\,\nabla\cdot\vec{v} = J\Big[\underbrace{\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}}\Big]$$

$$= J\,\dot{\varepsilon}_{ii} \qquad D_{ii} \approx \dot{\varepsilon}_{ii}$$