## Integration-by-parts

$$\int_a^b w \frac{dv}{dx}\,dx = -\int_a^b v \frac{dw}{dx}\,dx + w(b)v(b) - w(a)v(a)$$

We can establish this by:

$$\frac{d}{dx}(wv) = \frac{dw}{dx}v + w\frac{dv}{dx} \Rightarrow \boxed{w\frac{dv}{dx} = \frac{d}{dx}(wv) - \frac{dw}{dx}v}$$

$$\int_a^b w \frac{dv}{dx}\,dx = \int_a^b \left[\frac{d}{dx}(wv) - \frac{dw}{dy}v\right]dx$$

$$= \left[wv\right]_a^b - \int v\frac{dw}{dx}\,dx$$

Consider

$$\int_a^b w \frac{d^2u}{dx^2}\, dx = \int_a^b w \frac{d}{dx}\left(\frac{du}{dx}\right) dx = \int_a^b w \frac{dv}{dx}\, dx$$

$$\text{Let } v = \frac{du}{dx}$$

$$\int_a^b w \frac{d^2u}{dx^2}\, dx = -\int_a^b v \frac{dw}{dx}\, dx + w(b)\,v(b) - w(a)\,v(a)$$

$$= -\int_a^b \frac{du}{dx}\frac{dw}{dx}\, dx + w(b)\left.\frac{du}{dx}\right|_b - w(a)\left.\frac{\partial u}{\partial x}\right|_a$$

$$\boxed{-\int_a^b \frac{dw}{dx}\frac{du}{dx}\, dx = \int_a^b w \frac{d^2u}{dx^2}\, dx + w(a)\left.\frac{du}{dx}\right|_a - w(b)\left.\frac{du}{dx}\right|_b}$$

$$\Pi(u) = \int_0^L \frac{EA}{2}\left(\frac{du}{dx}\right)^2 dx - P\,u(L)$$

minimize w.r.t. $u(0) = 0$

$$\int G \cdot \eta \, dx = 0$$
$$\int G^2 \, dx = 0$$

$G$ $\quad$ $\mu$ $\quad$ $G = 0$

$$\delta \Pi(u) = \Pi(\delta u) = 0$$

$$= \int_0^L EA \frac{du}{dx}\frac{d(\delta u)}{dx}\, dx - P\,\delta u = \int_0^L \Big[\underbrace{-\frac{d}{dx}\left(EA\frac{du}{dx}\right)}_{G}\Big]\underbrace{\delta u}_{\mu}\, dx + \delta u\left[EA\frac{du}{dx}\right]_0^L - P\,\delta u(L) = 0$$

variational principle

$$\delta\left(\frac{du}{dx}\right) = \frac{d(\delta u)}{dx}$$

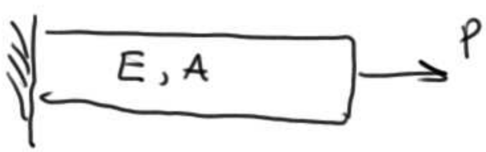
$$EA\frac{du}{dx}\Big|_0 \cancelto{0}{\delta u(0)}$$

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) = 0$$

$$EA\frac{du(L)}{dx} - P = 0$$ natural Neumann

$$u(0) = 0$$ essential Dirichlet

$$K\vec{u} = \vec{b}$$



$$\int_0^L F(u, u', x)\,dx \Rightarrow \quad \delta F = \int_0^L \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'\,dx$$

$$\int_0^L \frac{\partial F}{\partial u}\delta u - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\delta u\,dx - \left[\frac{\partial F}{\partial u'}\delta u\right]_0^L$$

$$\int_0^L \underbrace{\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]}_{\text{Euler-Lagrange Eqn.}}\delta u\,dx - \left[\frac{\partial F}{\partial u'}\delta u\right]_0^L$$

## Natural & Essential B.C.'s

$$I(u) = \int_a^b F(u, u', x)\,dx - Q_a u(a) - Q_b u(b)$$

For $I$ to be stationary

$$0 = \int_a^b \delta u \underbrace{\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]}_{\text{satisfied by D.E.}} dx + \underbrace{\left(\left.\frac{\partial F}{\partial u'}\right|_b - Q_b\right)\delta u(b) - \left(\left.\frac{\partial F}{\partial u'}\right|_a + Q_a\right)\delta u(a)}_{=0}$$

Any of following

1) $\delta u(a) = 0$, $\delta u(b) = 0$
2) $\delta u(a) = 0$, $\left.\frac{\partial F}{\partial u'}\right|_b - Q_b = 0$
3.) $-\left.\frac{\partial F}{\partial u'}\right|_a - Q_a = 0$, $\delta u(b) = 0$
4) $-\left.\frac{\partial F}{\partial u'}\right|_a - Q_a = 0$
$\left.\frac{\partial F}{\partial u'}\right|_b - Q_b = 0$

## Variational Formulations

Classically "variational formulation" refer to constructing a functional or a variational principle that is equivalent to the governing equation.

The modern use refers to a formulation where the governing equs. are translated into an equivalent weighted-integral statement

## Weighted Integral Statement

$$u \approx u^h = \sum_{j=1}^{n} N_j u_j + \sum_{j=1}^{m} \phi_j c_j$$

$u_j \Rightarrow$ "nodes", but we have no "nodes"

$$u \approx u^h = \sum_{j=1}^{m} c_j \phi_j + \phi_0$$ $\leftarrow$ sole purpose is to satisfy the B.C.'s