Consider the ODE $\rightarrow \underline{u \approx u^h = \sum_j c_j \phi_j + \phi_0}$

$$-\frac{d}{dx}\left[a(x) \frac{du}{dx} \right] + c(x) u = f(x) \qquad 0 < x < L$$

$$u(0) = u_0 \qquad \left[a(x) \frac{\partial u}{\partial x} \right]_{x=L} = Q_0$$

Let's choose

$$\begin{cases} \phi_0 = 1 \\ \phi_1 = x^2 - 2x \\ \phi_2 = x^3 - 3x \end{cases}$$

Let $L = 1 \quad u_0 = 1 \quad Q_0 = 0$

$a(x) = x \,,\; c(x) = 1 \,,\; f(x) = 0$

$$-\frac{d}{dx}\left[x \frac{du}{dx} \right] + u = 0 \Rightarrow \boxed{\begin{gathered} \frac{du}{dx} + x \frac{d^2u}{dx^2} + u = 0 \\ u(0) = 1 \quad + \quad x \frac{du}{dx}\Big|_{x=L} = 0 \end{gathered}}$$

$$u \approx u^h = c_1 (x^2 - 2x) + c_2 (x^3 - 3x) + 1$$

$$\frac{du}{dx} = c_1(2x-2) + c_2(3x^2-3)$$

$$\frac{d^2u}{dx} = 2c_1 + 6x\,c_2$$

$$-c_1(2x-2) - c_2(3x^2-3) - 2x\,c_1 - 6c_2x^2 + c_1(x^2-2x) + c_2(x^3-3x) + 1 = 0$$

x^3: $c_2 = 0$

x^2: $-3c_2 - 6c_2 + c_1 = -9c_2 + c_1 = 0$

x^1: $-2c_1 - 2c_1 - 2c_1 - 3c_2 = -6c_1 - 3c_2 = 0$

x^0: $2c_1 + 3c_2 + 1 = 0$

Go back

$$\delta u \left[-\frac{d}{dx}\left[x\frac{du}{dx} \right] + u \right] = [0]\,\delta u$$

$$\int_0^L \underbrace{\delta u}_{\omega} \underbrace{\left[-\frac{d}{dx}\left[x\frac{du}{dx} \right] + u \right]}_{R} dx = 0 \quad \Rightarrow \quad \int_0^L \omega R\, dx = 0$$

$$R = c_2 x^3 + (c_1 - 9c_2)x^2 + (-6c_1 - 3c_2)x + 2c_1 + 3c_2 + 1$$

choose $\delta u_1 = 1$, $\delta u_2 = x$

$$0 = \int_0^1 1 \cdot R\, dx = (1 + 2c_1 + 3c_2) + \frac{1}{2}(-6c_1 - 3c_2) + \frac{1}{3}(c_1 - 9c_2) + \frac{1}{4}c_2$$

$$0 = \int_0^1 x \cdot R\, dx = (1 + 2c_1 + 3c_2) + \frac{1}{3}(-6c_1 - 3c_2) + \frac{1}{4}(c_1 - 9c_2) + \frac{1}{5}c_2$$

$$c_1 = \frac{222}{23} \quad , \quad c_2 = \frac{-100}{23}$$

Depending on the choice of δu_i we arrive at the different weighted residual methods. If we choose δu_i

$$\delta u_i = \varphi_i \quad \Rightarrow \quad \text{Galerkin Method}$$

$$\delta u_i \neq \varphi_i \quad \Rightarrow \quad \text{Petrov-Galerkin Method}$$

$$\delta u_i = \frac{d}{dx}\left(a(x) \frac{d\varphi_i}{dx} \right) \Rightarrow \text{least-squares method}$$

$$\delta u_i = \Delta(x - x_i) \Rightarrow \text{collocation method}$$

where Δ is Dirac Delta function

$$\Delta = 0 \quad x \neq x_i$$

$$\Delta = 1 \quad x = x_i$$

Only L-S method results in a symm. coeff. matrix

$$\begin{bmatrix} 7/3 & -5/4 \\ -3/4 & -31/20 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} \leftarrow \text{not symm.}$$

Ritz or Rayleigh-Ritz Method

utilizes a quadric functional derived from the "weak form"

Weak Form

Step 1 Same as w-r

$$-\frac{d}{dx}\left[a(x)\frac{du}{dx}\right] = f(x) \qquad \text{for } 0 < x < L$$

subject to

$$u(0) = U_0 \qquad \left(a\frac{du}{dx}\right)\Big|_{x=L} = Q_L$$

$$\int_0^L \left\{-\frac{d}{dx}\left[a(x)\frac{du}{dx}\right] - f(x)\right\}\delta u\, dx = 0$$

Step 2

$$0 = \int_0^L \left\{a(x)\frac{d(\delta u)}{dx}\frac{du}{dx} - \delta u\, f(x)\right\}dx - \left[\delta u\, a(x)\frac{du}{dx}\right]_0^L$$

Step 3   Impose B.C.'s

$$0 = \int_0^L \left\{ a(x) \frac{d}{dx}(\delta u) \frac{du}{dx} - \delta u\, f(x) \right\} dx - \delta u\, Q_L$$

Weak Form → Variational Form

Bilinear Form

The weak form will contain 2 types of expressions, those involving δu + u and those involving only δu

Group

$$B(\delta u, u) = \int_0^L a(x) \frac{d}{dx}(\delta u) \frac{du}{dx}\, dx$$

$$l(\delta u) = \int_0^L \delta u\, f(x)\, dx + \delta u\, Q_L$$

We can write / the problem is stated as, find u:

$B(\delta u, u) = l(\delta u)$ "variational problem"

The functional $B(\delta u, u)$ is said to be <u>bilinear</u>

$$B(\alpha u_1 + \beta u_2, v) = \alpha B(u_1, v) + \beta B(u_2, v)$$

$$B(u, \alpha v_1 + \beta v_2) = \alpha B(u, v_1) + \beta B(u, v_2)$$

B is symm.

$$B(u, v) = B(v, u)$$

If bilinear & symm.

$$B(\delta u, u) = \frac{1}{2} \delta B(u, u) \quad , \quad \ell(\delta u) = \delta \ell(u)$$

$$B(\delta u, u) = \int_0^L EA \frac{d}{dx}(\delta u) \frac{du}{dx} = \delta \int_0^L \frac{EA}{2} \left(\frac{du}{dx}\right)^2 dx$$

$$B(\delta u, u) = l(\delta u) \Rightarrow B(\delta u, u) - l(\delta u) = 0$$

$$\frac{1}{2}\delta B(u,u) - \delta l(u) \equiv \delta I(u) = 0$$

Restate the variational problem

$$I(u) = \frac{1}{2} B(u,u) - l(u)$$