

$$B(\delta u, u) = l(\delta u)$$

$$I(u) = \frac{1}{2} B(u, u) - l(u)$$

Ritz Method

Use the "weak form". Has advantage the approximating functions (ϕ_i 's) only need to satisfy the essential B.C.'s, since the natural B.C.'s are included. We seek

$$u \approx u^h = \sum_{j=1}^n c_j \phi_j(x)$$

$I(u^h) \leftarrow$ sub. in u^h , integrate, $I(c_j)$

$$\frac{\partial I}{\partial c_j} = 0$$

Example

$$-\frac{\partial^2 u}{\partial x^2} + u + x^2 = 0 \quad \text{for } 0 < x < 1$$

$$\text{with } u(0) = 0, \quad u(1) = 0$$

$$\int_0^1 \underbrace{\left\{ \frac{d}{dx}(\delta u) \frac{du}{dx} - \delta u u \right\}}_{B(\delta u, u)} dx + \int_0^1 \underbrace{\delta u x^2}_{l(\delta u)} dx = 0$$

$$I(u) = \frac{1}{2} B(u, u) - l(u) = \frac{1}{2} \int_0^1 \left[\left(\frac{\partial u}{\partial x} \right)^2 - u^2 + 2x^2 u \right] dx$$

$$u \approx u^h = c_1 x(1-x) + c_2 x^2(1-x) + c_3 x^3(1-x)$$

$$\frac{\partial I}{\partial c_1} = 0$$

$$\frac{\partial I}{\partial c_2} = 0$$

$$\frac{\partial I}{\partial c_3} = 0$$

Interpolation functions?

Again $u \approx u^h = C_j \phi_j + \phi_0$

ϕ_0 satisfies essential B.C.'s

otherwise the ϕ_j have to satisfy the following

- 1.) ϕ_j must be selected such that $B(\phi_i, \phi_j)$ is defined and nonzero
i.e. they must have proper continuity

$$\frac{\partial^2 u}{\partial x^2} \quad u^h = x$$

ϕ_j must satisfy the homogeneous form of the specified B.C.'s

i.e. $u(0) = u_0$ ϕ_j must satisfy $u(0) = 0$

- 2.) The set of $\{\phi_j\}$ be linearly independent
 $\phi_1 = x(1-x)$ $\phi_2 = x^2(1+x)$ $\phi_3 = 2x^2(1-x)$

The set $\{\phi_i\}$'s must be complete

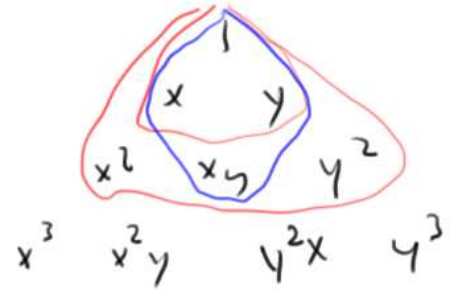
$$\{x, x^2, x^3, x^4\} \rightarrow \text{complete}$$

$$\{x, y, xy, x^2y, xy^2, x^2y^2\} \rightarrow \text{complete}$$

$$\{x^3, x^5, x^{25}\} \rightarrow \text{NOT complete}$$

$$\{x, x^2, xy^3\} \rightarrow \text{NOT complete}$$

Pascal's Triangle



4.) ϕ_0 must be the lowest order function that satisfies the B.C.'s

Almost always use polynomials.

Example

$$-\frac{\partial^2 u}{\partial x^2} + u + x^2 = 0 \quad \text{for } 0 < x < 1$$

$$u(0) = 0, \quad u(1) = 0$$

$$I(u) = \frac{1}{2} \int_0^1 \left[\left(\frac{du}{dx} \right)^2 - u^2 + 2x^2 u \right] dx$$

$$u \approx u^h = C_j \phi_j = C_1 + C_2 x + C_3 x^2$$

$$u(0) = 0 = C_1$$

$$u(1) = 0 = C_2 + C_3 = 0 \quad \Rightarrow \quad C_2 = -C_3$$

$$u^h = -C_3 x + C_3 x^2 = C_3 (x^2 - x)$$