$$B(\delta u, u) = \ell(\delta u)$$

$$I(u) = \frac{1}{2} B(u,u) - \ell(u)$$

Ritz Method

Use the "weak form". Has advantage the approximating functions (ϕ_i's) only need to satisfy the essential B.C.'s, since the natural B.C.'s are included. We seek

$$u \approx u^h = \sum_{j=1}^{n} c_j \phi_j(x)$$

$I(u^h) \leftarrow$ sub. in u^h, integrate, $I(c_j)$

$$\frac{\partial I}{\partial c_j} = 0$$

Example

$$-\frac{\partial^2 u}{\partial x^2} + u + x^2 = 0 \qquad \text{for} \quad 0 < x < 1$$

with $\quad u(0) = 0 \quad , \quad u(1) = 0$

$$\int_0^1 \underbrace{\left\{ \frac{d}{dx}(\delta u)\, \frac{du}{dx} - \delta u \; u \right\}}_{B(\delta u,\, u)} dx + \int_0^1 \underbrace{\delta u \; x^2 \, dx}_{l(\delta u)} = 0$$

$$I(u) = \frac{1}{2} B(u,u) - l(u) = \frac{1}{2} \int_0^1 \left[\left(\frac{\partial u}{\partial x} \right)^2 - u^2 + 2x^2 u \right] dx$$

$$u \approx u^h = c_1\, x(1-x) + c_2\, x^2 (1-x) + c_3\, x^3 (1-x)$$

$$\frac{\partial I}{\partial c_1} = 0 \qquad \frac{\partial I}{\partial c_2} = 0 \qquad \frac{\partial I}{\partial c_3} = 0$$

Interpolation functions ?

Again $u \simeq u^h = c_j \phi_j + \phi_0$

ϕ_0 satisfies essential B.C.'s

Otherwise the ϕ_i have to satisfy the following

1.) ϕ_j must selected such that $B(\phi_i, \phi_j)$ it's defined and nonzero

i.e. they must have proper continuity

$\frac{\partial^2 u}{\partial x^2}$ $u^h = x$

ϕ_j must satisfy the homogenuous form of the specified B.C.'s

i.e. $u(0) = u_0$ ϕ_j must satisfy $u(0) = 0$

2.) The set of $\{\phi_j\}$ be linealy independent

$\phi_1 = x(1-x)$ $\phi_2 = x^2(1+x)$ $\phi_3 = \cancel{2x^2}(1-x)$

The set $\{\phi_i\}$'s must be <u>complete</u>

$\{x, x^2, x^3, x^4\} \to$ complete

$\{x, y, xy, x^2y, xy^2, x^2y^2\} \to$ complete

$\{x^3, x^5, x^{25}\} \to$ NOT complete

$\{x, x^2, xy^3\} \to$ NOT complete

4.) ϕ_0 must be the lowest order function that satisfies the B.C.'s

Almost alway use polynomials

Pascal's Triangle

1

$x \quad y$

$x^2 \quad xy \quad y^2$

$x^3 \quad x^2y \quad y^2x \quad y^3$

Example

$$-\frac{\partial^2 u}{\partial x^2} + u + x^2 = 0 \qquad \text{for} \quad 0 < x < 1$$

$$u(0) = 0, \qquad u(1) = 0$$

$$I(u) = \frac{1}{2}\int_0^1 \left(\frac{du}{dx}\right)^2 - u^2 + 2x^2 u\Big]dx$$

$$u \approx u^h = c_j \phi_j = c_1 + c_2 x + c_3 x^2$$

$$u(0) = 0 = c_1$$

$$u(1) = 0 = c_2 + c_3 = 0 \Rightarrow c_2 = -c_3$$

$$u^h = -c_3 x + c_3 x^2 = c_3 (x^2 - x)$$