$$-\frac{d}{dx}\left(a\frac{du}{dx}\right) + cu - f = 0 \qquad \text{for} \quad a < x < b$$

Subject to Neumann B.C.'s

$$\left(a\frac{\partial u}{\partial x}\right)_{x=a} = Q_a \quad + \quad \left(a\frac{\partial u}{\partial x}\right)\Big|_{x=b} = Q_b$$

<u>Weak form</u>

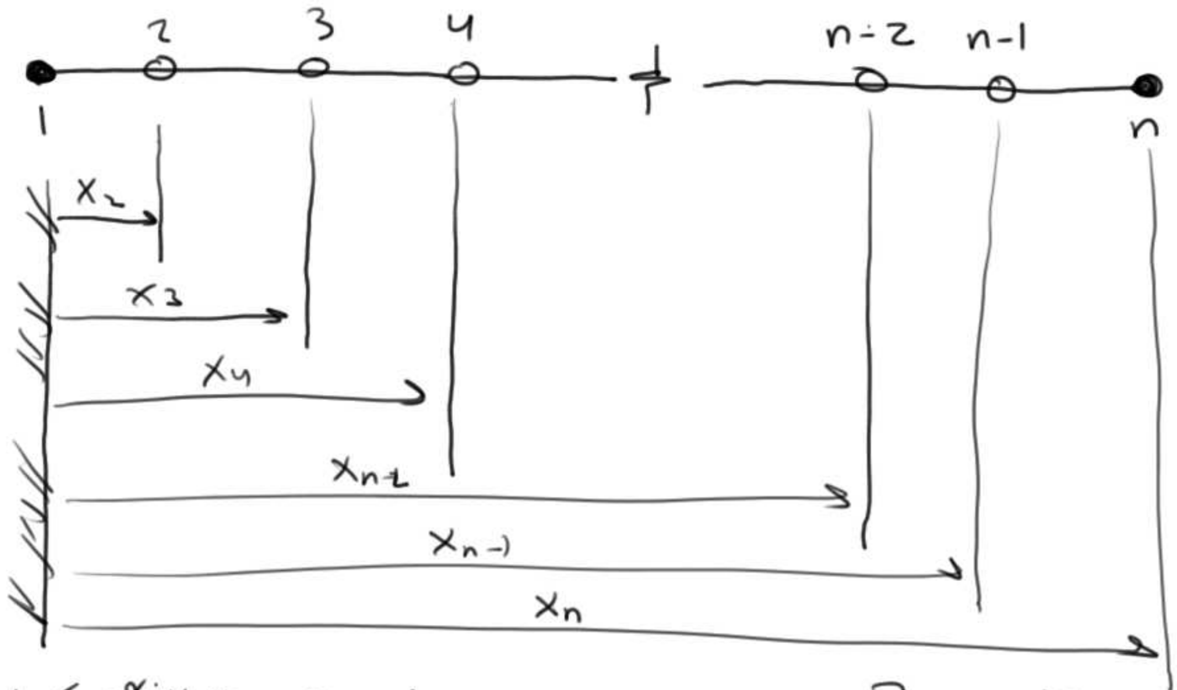
$$\int_a^b \left[a\frac{\partial \delta u}{\partial x}\frac{\partial u}{\partial x} + c\,\delta u\, u - \delta u\, f\right] dx - \delta u(a) Q_a - \delta u(b) Q_b$$

$$B(\delta u, u) = \int_a^b \left[a\frac{d\delta u}{\partial x}\frac{du}{\partial x} + c\,\delta u\, u\right] dx$$

$$\ell(\delta u) = \int_a^b \delta u\, f\, dx + \delta u(a) Q_a + \delta u(b) Q_b$$

$$B(\delta u, u) = \ell(\delta u)$$

$u \approx u^h = \sum_{j=1}^{n} N_j u_j$ $\quad N_j$ of degree $n-1$ + u_j are unknowns



$$0 = \sum_{i=1}^{n-1} \left\{ \int_{x_i}^{x_{i+1}} \left[a \frac{d\delta u}{\partial x} \frac{du}{dx} + \delta u \, u - \delta u \, f \right] dx - \left[\delta u(x) \, a \frac{du}{dx} \right]_{x_i}^{x_{i+1}} \right\}$$

$$= \int_{x_1}^{x_n} \left[a \frac{d\delta u}{dx} \frac{du}{dx} + c \, \delta u \, u - \delta u \, f \right] dx - \delta u(x_1) \left[-a \frac{du}{dx} \right]_{x_1} - \delta u(x_2) \left[a \frac{du}{dx} \right]_{x_2}$$

$$- \delta u(x_2) \left[-a \frac{du}{dx} \right]_{x_2} - \delta u(x_3) \left[a \frac{du}{dx} \right]_{x_3} - \dots - \delta(x_{n-1}) \left[a \frac{du}{dx} \right]_{x_{n-1}} - \delta u(x_n) \left[a \frac{du}{\partial x} \right]_{x_n}$$

$$0 = \int_{x_1}^{x_n} \left[a \frac{d\delta u}{dx}\frac{du}{dx} + c\,\delta u\, u - \delta u\, f \right] dx - \delta u(x_1) Q_1 - \delta u(x_2) Q_2 - \dots - \delta u(x_{n-1}) Q_{n-1} - \delta u(x_n) Q_n$$

where

$$Q_1 = \left[-a \frac{du}{dx} \right]_{x_1}$$

$$\left.\begin{aligned} Q_2 &= \left[\left(a\frac{du}{dx}\right)_{x_2^-} - \left(a\frac{du}{dx}\right)_{x_2^+} \right] \\ &\vdots \\ Q_{n-1} &= \left[\left(a\frac{du}{dx}\right)_{x_{n-1}^-} - \left(a\frac{du}{dx}\right)_{x_{n-1}^+} \right] \end{aligned}\right\} = 0$$

$$Q_n = \left[a \frac{du}{dx} \right]_n$$

$$0 = \int_{x_a}^{x_b} \left(a \frac{d\delta u}{dx}\frac{du}{dx} + c\,\delta u\, u \right) dx - \int_{x_a}^{x_b} \delta u\, f\, dx - \sum_{j=1}^{n} \delta u(x_i) Q_j$$

Let $u = N_j u_j$ & $\delta u = N_i$

For $i = 1$

$$0 = \int_{x_a}^{x_b} \left[a \frac{dN_1}{dx} \left(\frac{d}{dx} (N_j u_j) \right) + c\, N_1 (N_j u_j) \right] dx - \int_{x_a}^{x_b} N_1 f \, dx - \sum_{j=1}^{n} N_1(x_j) Q_j$$

$\vdots$

$$0 = \int_{x_a}^{x_b} \left[c \frac{dN_n}{dx} \frac{dN_j}{dx} u_j + c\, N_1 N_j u_j \right] dx - \int_{x_a}^{x_b} N_n f \, dx - \sum_{j=1}^{n} N_n(x_j) Q_i$$

$\therefore$ i^{tn} equation

$$0 = \underbrace{\int_{x_a}^{x_b} \left[a \frac{dN_i}{dx} \frac{dN_j}{dx} + c\, N_i N_j \right] dx}\; u_j - \underbrace{\int_{x_a}^{x_b} N_i f \, dx}_{f_i} - Q_i$$

$$\underbrace{B(N_i, N_j)}\, u_j \underbrace{- f_i - Q_i}_{F_i} = 0 \qquad N_i(x_j) = \begin{cases} i = j \rightarrow 1 \\ i \neq j \rightarrow 0 \end{cases}$$

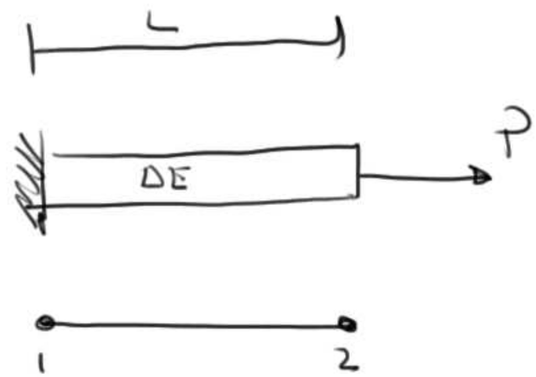
$$K_{ij} u_j = F_i \qquad \vec{u} = \overleftrightarrow{K}^{-1} \vec{F}$$

Example

Let $(x_a, x_b) = (0, L)$

$c = 0$

$a = EA$



$$K_{ij} = \int_0^L \left(EA \frac{dN_i}{dx} \frac{dN_j}{dx} \right) dx$$

$$\begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} \\ -\frac{AE}{L} & \frac{AE}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$