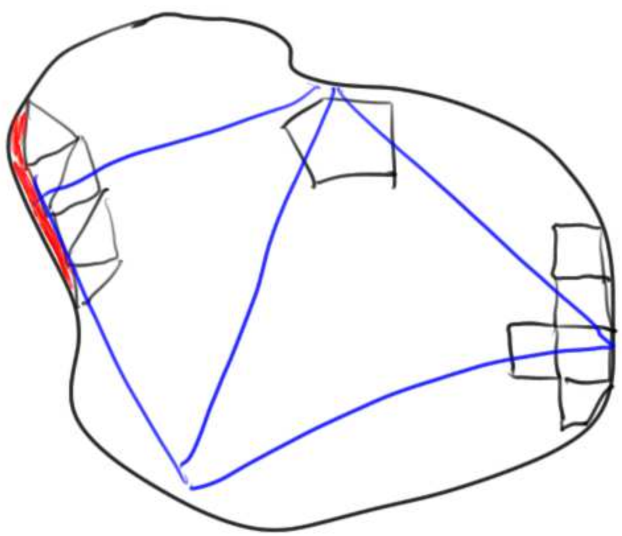
2D problems, → scalar field variables, u

— no mesh error



General Rules

1. Element should be able to reproduce fields on the order of the governing equation
2. #, shape, type → accurate
3. The mesh density should cover areas of high gradients

Element type
Triangles
Quads

elements
→ size

Degree of accuracy - interpoland

— no magic formula

* Grade away gradually

$$-\underbrace{\frac{\partial}{\partial x}\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right)}_{\nabla\cdot(A\nabla u)} + a_{00}u - f = 0$$

$u = u(x,y)$ $\quad a_{ij}$ are the data $\quad$ with B.C.'s

<u>Models</u>

Heat transfer
Irrotational flow of Ideal Fluid
Groundwater flow through permeable geology

Weak Form

$$0 = \int_{\Omega} \delta u \left[-\frac{\partial}{\partial x}(F_1) - \frac{\partial}{\partial y}(F_2) + a_{00}u - f \right] dx\,dy$$

where

$$F_1 = a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}, \quad F_2 = a_{22}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}$$

Integrat - by - parts

$$\frac{\partial}{\partial x}\left(\delta u\, F_1\right) = \frac{\partial u}{\partial x} F_1 + \delta u \frac{\partial F_1}{\partial x} \Rightarrow -\delta u \frac{\partial F_1}{\partial x} = \frac{\partial \delta u}{\partial x} F_1 - \frac{\partial}{\partial x}\left(\delta u\, F_1\right)$$

$$\frac{\partial}{\partial y}\left(\delta u\, F_2\right) = \frac{\partial u}{\partial y} F_2 + \delta u \frac{\partial F_2}{\partial x} \Rightarrow -\delta u \frac{\partial F_2}{\partial x} = \frac{\partial \delta u}{\partial y} F_2 - \underbrace{\frac{\partial}{\partial y}\left(\delta u\, F_2\right)}$$

Divergence Theorem

$$\int_\Omega \frac{\partial}{\partial x}\left(\delta u\, F_1\right) dx\,dy = \oint_\Gamma \delta u\, F_1\, n_x\, dS_\Gamma$$

$$\int_\Omega \frac{\partial}{\partial x}\left(\delta u\, F_2\right) dx\,dy = \oint_\Gamma \delta u\, F_2\, n_y\, dS_\Gamma$$

n_x & n_y are components of the unit normals to Γ

$\hat{n} = n_x \hat{e}_x + n_y \hat{e}_y$ on Γ

$$0 = \int_\Omega \left[\frac{\partial \delta u}{\partial x}\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) + \frac{\partial \delta u}{\partial y}\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right) + a_{00}\,\delta u\, u - \delta u\, f \right] dx\,dy$$

$$- \oint_\Gamma \delta u \left[n_x\left(a_{11}\frac{\partial u}{\partial x} + a_{12}\frac{\partial u}{\partial y}\right) + n_y\left(a_{21}\frac{\partial u}{\partial x} + a_{22}\frac{\partial u}{\partial y}\right)\right) dS \quad \equiv q^n$$

$$0 = \int_\Omega \left[\frac{\partial \delta u}{\partial x}\left(a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y} \right) + \frac{\partial \delta u}{\partial y}\left(a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y} \right) + a_{00}\, \delta u\, u - \delta u\, f \right] dx\,dy$$

$$- \oint_\Gamma \delta u\, q_n\, dS \quad ///$$

$$B(\delta u, u) = \int_\Omega \left[\frac{\partial \delta u}{\partial x}\left(a_{11} \frac{\partial u}{\partial x} + a_{12} \frac{\partial u}{\partial y} \right) + \frac{\partial \delta u}{\partial y}\left(a_{21} \frac{\partial u}{\partial x} + a_{22} \frac{\partial u}{\partial y} \right) + a_{00}\, \delta u\, u \right] dx\,dy$$

$$l(\delta u) = \int_\Omega \delta u\, f\, dx\,dy + \oint_\Gamma \delta u\, q_n\, dS$$

$$B(\delta u, u) = l(\delta u)$$

B is only symm. if $a_{12} = a_{21}$

Let

$$C = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{00} \end{bmatrix}, \quad D() = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ 1 \end{bmatrix} u \quad \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ u \end{bmatrix}$$

$$B(\delta u, u) = \int_\Omega \begin{bmatrix} \frac{\partial \delta u}{\partial x} \\ \frac{\partial \delta u}{\partial y} \\ \delta u \end{bmatrix}^T \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{00} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ 1 \end{bmatrix} dx\,dy$$

$$\left.\begin{aligned} B(\delta u, u) &= \int_\Omega (D\,\delta u)^T C\,(D u)\,dx\,dy \\ \ell(\delta u) &= \int_\Omega \{\delta u\}^T \vec{f}\,dx\,dy + \int_\Gamma \delta u^T \hat{q}_n\,dS \end{aligned}\right\} \rightarrow \text{Weak form}$$

FE Model

$$u \approx u^h(x,y) = N_j u_j \qquad \delta u_i = N_i$$

$$\Big\{ \overbrace{\int_\Omega \left[\frac{\partial N_i}{\partial x}\left(a_{11}\frac{\partial N_j}{\partial x} + a_{21}\frac{\partial N_j}{\partial y}\right) + \frac{\partial N_i}{\partial y}\left(a_{21}\frac{\partial N_j}{\partial x} + a_{22}\frac{\partial N_j}{\partial y}\right) + a_{00} N_i N_j \right] dx\,dy}^{K_{ij}}\; u_j$$

$$- \underbrace{\int_\Omega N_i f\,dx\,dy}_{f_i} - \underbrace{\oint_\Gamma N_i q_n\,dS}_{Q_i} \Big\} u_j = 0 \qquad K_{ij} u_j = f_i + Q_i$$

$$\vec{K}\vec{u} = \vec{F} = \vec{f} + \vec{Q}$$

where

$$K = \int B^T C\, B\, dx\,dy \quad , \quad \vec{f} = \int N^T \vec{f}\, dy\,dx \quad , \quad \vec{Q} = \oint N^T \vec{q}_n\, dS$$

$$B = D\, N^T \begin{bmatrix} N_{1,x} & N_{2,x} & , \dots & N_{n,x} \\ N_{1,y} & N_{2,y} & , \dots & N_{n,y} \\ N_1 & N_2 & , \dots & N_n \end{bmatrix}$$