Plane elasticity

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \rho b_1 = \rho \frac{\partial^2 u_1}{\partial t^2} \Rightarrow \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \rho b_x = \rho \frac{\partial^2 u_x}{\partial t^2}$$

$$\frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{22}}{\partial x_2} + \rho b_2 = \rho \frac{\partial^2 u_2}{\partial t^2} \Rightarrow \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \rho b_y = \rho \frac{\partial^2 u_y}{\partial t^2}$$

$$\bar{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \qquad D^T = \begin{bmatrix} \partial/\partial x & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \end{bmatrix}$$

$$D^T \bar{\sigma} + \rho \vec{b} = \rho \ddot{\vec{u}} \qquad \vec{b} = \begin{Bmatrix} b_x \\ b_y \end{Bmatrix} \qquad \vec{u} = \begin{Bmatrix} u_x \\ u_y \end{Bmatrix}$$

$$\vec{\varepsilon} = D \vec{u} \qquad \vec{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix}$$

$$\vec{\sigma} = \underset{\sim}{\vec{C}}\,\vec{\varepsilon}$$

$$D^T \vec{\sigma} = -\rho \vec{b} + \rho \ddot{\vec{u}}$$

$$D^T C \vec{\varepsilon} = \quad '' \qquad ''$$

$$D^T C D \vec{u} = \quad '' \qquad ''$$

Natural B.C.s

$$\left.\begin{aligned} t_x &= \sigma_{xx}\hat{n}_x + \sigma_{xy}\hat{n}_y \\ t_y &= \sigma_{xy}\hat{n}_x + \sigma_{yy}\hat{n}_y \end{aligned}\right\} \text{ on } \Gamma_\sigma \qquad \vec{t} = \vec{\sigma}\,\hat{n}$$

Essential B.C.s

$u_x = u_{0x}$, $u_y = u_{0y}$ on Γ_u

$$0 = \int_{V_e} (\sigma_{ij}\, \delta\varepsilon_{ij} + \rho\, \ddot{u}_i\, \delta u_i)\, dV_e - \int_{V_e} b_i \delta u_i\, dV - \oint t_i\, \delta u_i\, dS$$

$$V_e = h_e\, \Omega_e$$

$$0 = \int_{\Omega} h_e \left[\sigma_{xx}\, \delta\varepsilon_{xx} + \sigma_{yy}\, \delta\varepsilon_{yy} + 2\, \sigma_{xy}\, \delta\varepsilon_{xy} + \rho (\ddot{u}_x\, \delta u_x + \ddot{u}_y\, \delta u_y) \right] dx\, dy$$
$$- \int_{\Omega} h_e (b_x\, \delta u_x + b_y\, \delta u_y)\, dx\, dy - \oint_{\Gamma_e} h_e (t_x\, \delta u_x + t_y\, \delta u_y)\, dS$$

$$0 = \int_{\Omega} h_e \left[(D\, \delta\vec{u})^T C\, (D\, \vec{u}) + \rho\, \delta\vec{u}^T \ddot{\vec{u}} \right] d\vec{x}$$
$$- \int_{\Omega_e} h_e (\delta\vec{u})^T \rho\, \vec{b}\, d\hat{x} - \oint_{\Gamma} h_e (\delta\vec{u})^T \vec{t}\, dS$$

$$\vec{u}^h = \begin{Bmatrix} u_x^h \\ u_y^n \end{Bmatrix} = \begin{Bmatrix} N_j \, u_{xj} \\ N_j \, u_{yj} \end{Bmatrix} = [N]\,\vec{d}$$

$$[N] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & \cdots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & \cdots & 0 & N_n \end{bmatrix}$$

$$\vec{d} = \left\{ u_x^1 \;\; u_y^1 \;\; u_x^2 \;\; u_y^2 \,, \ldots , \; u_x^n \,, u_y^n \right]^T$$

$$\delta\vec{u} = [N]$$

$$\vec{\varepsilon} = D\,\vec{u}^h = \underbrace{D[N]}\,\vec{d}$$

$B \rightarrow$ strain displacement matrix

$$0 = \Big[\underbrace{\int_\Omega h_e \, B^T C B \, d\vec{x}}_{K_e}\Big]\vec{d} + \Big[\underbrace{\int \rho [N]^T [N]}_{M_e}\Big] d\vec{x} \, \ddot{\vec{d}}$$

$$- \underbrace{\int_\Omega h_e \rho [N]^T \vec{b}}_{f^e} \, d\vec{x} - \oint_\Gamma \underbrace{h_e [N]^T \vec{t} \, dS}_{Q}$$

$$[k_e]\vec{d} + [\cancelto{0}{M_e}]\ddot{\vec{d}} - f - Q = 0$$

$$[k_e]d = F = f + Q$$

Ex