Mixed variational problem

$$a(u, \delta u) + b(\delta u, p) = f(\delta u)$$

$$b(\underline{u}, \delta p) = g(\delta p)$$

a & b are bilinear operators

energy

constraint → "saddle point" problems

$b \Rightarrow Q$ matrix

1st equation: $\left[\int B^T \alpha \, \vec{m} \, N^p \, d\vec{x} \right] p$

$\delta u = B^T = [\nabla \cdot N^u]^T$, $p = N^p_i p_i$

2nd equation: $\left[\underbrace{\int B^T \alpha \, \vec{m} \, N^p \, d\vec{x}}_{Q} \right] \dot{u}$

$\delta p = N^p$, $u = B^T \dot{u} = [\nabla \cdot N^u] \dot{u}$

Inf - sup condition

Babuška - Brezzi condition (BB) condition

Ladyzenkaja - BB condition

$$0 < \gamma \leq \inf_{p \neq 0} \sup_{u \neq 0} \frac{|b(u,p)|}{\|u\| \, \|p\|}$$

Proving inf-sup is sufficient for solvability & uniqueness

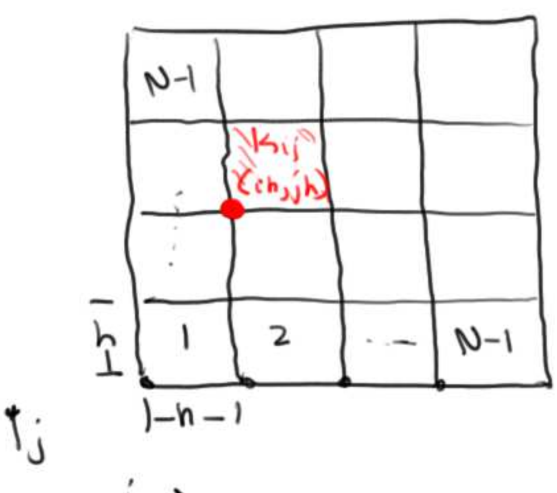
Discrete inf-sup

$$0 < \gamma^h \le \inf_{p^h \in M^h \setminus \{0\}} \sup_{u^h \in X^h \setminus \{0\}} \frac{|b(u^h, p^h)|}{\|u^h\|_{X^h} \|p^h\|_{M^h}}$$

equil. to

$$0 < \gamma \|p^h\|_{M^h} \le \sup_{u^h \in X^h \setminus \{0\}} \frac{|b(u^h, p^h)|}{\|u^h\|_X}$$

Consider



$\Omega = [0,1]^2$

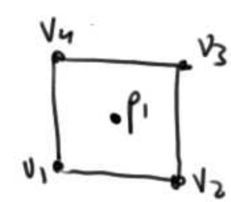
$h = \frac{1}{N}$ with N even

K_{ij} is the domain of the element w/ lower-left coordinates at (ih, jh)

$$\int_{K_{ij}} (\nabla \cdot v^h)\, p^h \, dx\, dy$$

$p^h \rightarrow$ piecewise constant

$v^h \rightarrow$ piecewise linear

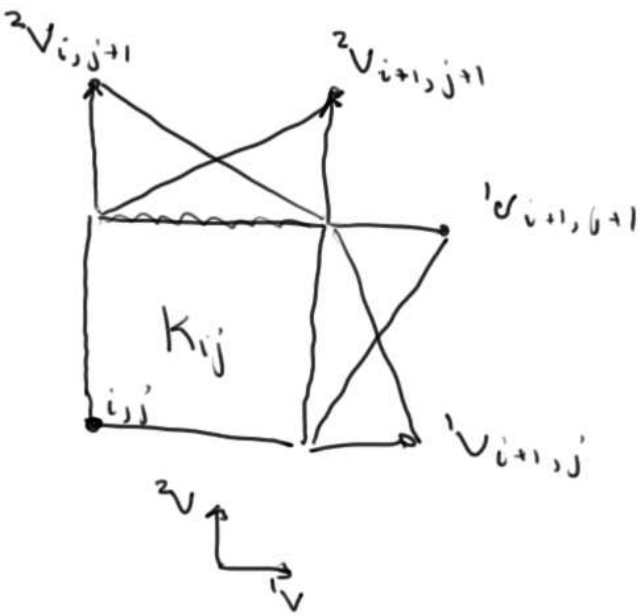


$$\int_{K_{ij}} (\nabla \cdot v^h)\, p^h \,dx\,dy = p_{i+\frac{1}{2}, j+\frac{1}{2}} \int_{\partial K_{ij}} \vec{v}^h \hat{n} \,dx\,dy$$

$$\vec{v} = \begin{Bmatrix} {}^1v \\ {}^2v \end{Bmatrix}$$

$$= p_{i+\frac{1}{2}, j+\frac{1}{2}} \Big({}^1v_{i+1,j} + {}^1v_{i+1,j+1} + {}^2v_{i+1,j+1} + {}^2v_{i+1,j+1} - {}^1v_{i,j} - {}^1v_{i,j+1}$$

$$- {}^2v_{i,j} - {}^2v_{i+1,j} \Big)$$



$$b(v^h, p^h) = \int_\Omega (\nabla \cdot v^h)\, p^h \,dx\,dy = \sum_{i,j=1}^{N-1} \int (\nabla \cdot v^h)\, p^h \,dx\,dy$$

$$0 = h^2 \sum_{i,j=1}^{N-1} {}^1v_{ij} (\partial_1^h p^h)_{ij} + {}^2v (\partial_2^h p^h)_{ij}$$

$$(\partial_1 p^h)_{ij} = \frac{1}{2h} \overbrace{\Big(p^h_{i+\frac{1}{2}, j+\frac{1}{2}} + p^h_{i+\frac{1}{2}, j-\frac{1}{2}} - p^h_{i-\frac{1}{2}, j+\frac{1}{2}} - p^h_{i-\frac{1}{2}, j-\frac{1}{2}} \Big)}^{0 =}$$

$$(\partial_2 p^h)_{ij} = \frac{1}{2h} \overbrace{\Big(p^h_{i+\frac{1}{2}, j+\frac{1}{2}} + p^h_{i-\frac{1}{2}, j+\frac{1}{2}} - p^h_{i+\frac{1}{2}, j-\frac{1}{2}} - p^h_{i-\frac{1}{2}, j-\frac{1}{2}} \Big)}^{0 =}$$

$$p_{i+\frac{1}{2},\, j+\frac{1}{2}} = p_{i-\frac{1}{2},\, j-\frac{1}{2}}$$
$$p_{i-\frac{1}{2},\, j+\frac{1}{2}} = p_{i+\frac{1}{2},\, j-\frac{1}{2}}$$

$$\left. p^h \right|_{K_{ij}} = (-1)^{i+j}$$

-1	1	-1	1
1	-1	1	-1
-1	1	-1	1
1	-1	1	-1

checkerboard pattern

"spurious" pressure mode

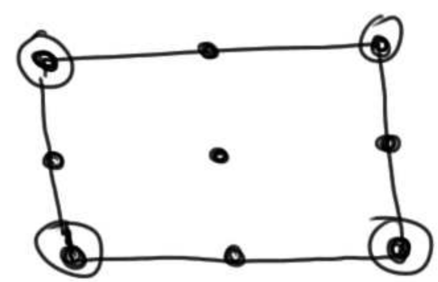
~~$u^h \rightarrow$ linears~~

~~$p^h \rightarrow$ linears~~

inf-sup

$u^h \rightarrow$ 9 node bi-quad

$p^h \rightarrow$ bilinear



• → disp

O → pressure