Time dependent problems

(a) coupled

$$u(\vec{x},t) \approx u^h(\vec{x},t) = u_j \, N_j(\vec{x},t)$$

(b) decoupled, i.e. seperable

$$u(\vec{x},t) \approx u^h(\vec{x},t) = u_j(t) \, N_j(\vec{x})$$

$$u(x,t) = T(t) \, X(x)$$

1. Spatial approximate w/ FEM

2. Temporal approx. w/ finite diffences

$$\frac{du}{dt} \approx \frac{\Delta u}{\Delta t}$$

$$\frac{\partial x}{\partial t} = \dot{x} = f(x,t)$$

$$t = t_0,\ t_0 + \Delta t,\ t_0 + 2\Delta t$$

$$\dot{x} = f(x_n, t_n)$$

$$\dot{x} = \frac{\Delta x}{\Delta t} = \frac{(x_{n+1} - x_n)}{\Delta t} = f(t_n, x_n)$$

$$x_{n+1} = x_n + \Delta t\, f(t_n, x_n)$$

$$x_0 = c$$

$$x_1 = \Delta t\, f(t_0, c) + c$$

$$x_2 = \Delta t\, f(t_1, x_1) + x_1$$

$$\vdots$$

$\dot{x} = \lambda x(t)$ $\qquad$ $x(0) = c$

$$x(t) = c e^{\lambda t}$$

$$R(\lambda) \leq 0$$

$$x_{n+1} = \lambda \Delta t \, x_n + x_n$$

$$= (1 + \lambda \Delta t) x_n$$

$$\vdots$$

$$= (1 + \lambda \Delta t)^2 x_{n-1}$$

$$\vdots$$

$$= (1 + \lambda \Delta t)^{n+1} x_0$$

$$|1 + \lambda \Delta t| \leq 1$$

$$\Delta t \leq \frac{2}{|\lambda|}$$

$$c = \sqrt{\frac{\varepsilon}{\rho}}$$

Implicit Method

$$\dot{x} = \frac{(x_n - x_{n-1})}{\Delta t}$$

$$\frac{(x_n - x_{n-1})}{\Delta t} = f(x_n, t_n)$$

$$x_n = \Delta t\, f(x_n, t_n) + x_{n-1}$$

$$x_{n+1} = \Delta t\, f(x_{n+1}, t_{n+1}) + x_n$$

$$\dot{x} = \lambda x(t)$$

$$x_{n+1} = \lambda \Delta t\, x_{n+1} + x$$

$$x_{n+1} - \lambda \Delta t\, x_{n+1} = x_n$$

$$(1 - \lambda \Delta t)\, x_{n+1} = x_n$$

$$x_{n+1} = \frac{x_n}{(1-\lambda\Delta t)^2} \qquad |1 - \lambda \Delta t| \geq 1$$

$$\vdots \quad \frac{x_0}{(1-\lambda\Delta t)^{n+1}}$$

Explicit Analysis

$$[M]\ddot{u} + \overbrace{[K]\vec{u}}^{f^{int}} = \vec{F} \Rightarrow [M]\ddot{\vec{u}} = f^{ext} - f^{int}$$

where $\boxed{[M] = \int \rho N_i N_j \, d\Omega}$

$$[K] = \int B^T C B \, d\Omega$$

$$\vec{F} = \vec{f} + \vec{Q} = \underbrace{\int N_i f_i \, d\Omega - \oint N_i t_i \, dS}_{f^{ext}}$$

Central Difference

Assume $\Delta t^{n+\frac{1}{2}} = t^{n+1} - t^n$

$$t^{n+\frac{1}{2}} = \frac{1}{2}(t^{n+1} + t^n)$$

$$\Delta t^n = t^{n+\frac{1}{2}} - t^{n-\frac{1}{2}}$$

$$\dot{u}^{n+\frac{1}{2}} = v^{n+\frac{1}{2}} = \frac{u^{n+1} - u^n}{\Delta t^{n+\frac{1}{2}}}$$

$$\ddot{u}^n = a^n = \left(\frac{v^{n+\frac{1}{2}} - v^{n-\frac{1}{2}}}{t^{n+\frac{1}{2}} - t^{n-\frac{1}{2}}}\right) = \frac{v^{n+\frac{1}{2}} - v^{n-\frac{1}{2}}}{\Delta t^n}$$

$$\boxed{\ddot{u}^n = \frac{\Delta t^{n-\frac{1}{2}}(u^{n+1} - u^n) - \Delta t^{n+\frac{1}{2}}(u^n - u^{n-1})}{\Delta t^{n+\frac{1}{2}} \Delta t^n \Delta t^{n-\frac{1}{2}}}}$$

if Δt is const.

$$= \frac{u^{n+1} - 2u^n + u^{n-1}}{\Delta t^2}$$

Lumped mass matrix

$$M^D_{ii} = \sum_j M^c_{ij}$$

or

$$M^D_{ii} = \sum_j M^c_{ij} = \int_\Omega \rho N_i \left(\sum_j N_j\right) dV = \int \rho N_I \, dV$$

Flowchart for Explict analysis

1. Initialize v^0; σ^0; u^0, $n=0$, $t=0$ $\rightarrow$ Compute $[M]$
2. Compute $f_0 = f^{ext} - f^{int}$
3. Compute $a^n = [M]^{-1} f^0$
4. Upate $t^{n+1} = t^n + \Delta t^{n+1/2}$, $t^{n+1/2} = \frac{1}{2}(t^n + t^{n-1})$
5. Upate $v^{n+1/2} = v^n + (t^{n+1/2} - t^n)\, a^n$
6. Enforce B.C.'s on $v^{n+1/2}$
7. Compute f^{n+1}
8. Compute a^{n+1}
9. Upate $v^{n+1} = v^{n+1/2} + (t^{n+1} - t^{n+1/2})\, a^{n+1}$
10. Check energy balance K.E. $\frac{1}{2}[m]\, \vec{v}^{\,n+1}\, \vec{v}^{\,n+1}$