$$\phi c_t \frac{\partial p(\vec{x})}{\partial t} = \nabla \cdot \left(\frac{\bar{\bar{k}}}{\mu} \nabla p(\vec{x}) \right)$$

$$\int_\Omega \omega(\vec{x}) \phi c_t \frac{\partial p(\vec{x})}{\partial t} = \nabla \cdot \left(\frac{\bar{\bar{k}}}{\mu} \nabla p(\vec{x}) \right) d\vec{x}$$

$$\int_\Omega \int_d \omega(x,y) \phi(x,y) c_t \frac{\partial p(x,y)}{\partial t} - \omega(x,y) \nabla \cdot \left(\frac{\bar{\bar{k}}}{\mu} \nabla p(x,y) \right) dz\, dA = 0$$

depth

$$\int_\Omega \omega(x,y) \underset{\downarrow}{d(x,y)} \phi c_t \frac{\partial p(x,y)}{\partial t} - \omega(x,y)\, d(x,y) \nabla \cdot \left(\frac{\bar{\bar{k}}}{\mu} \nabla p(x,y) \right) dA = 0$$

$$\sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} \left\{ \int_{x_i - \Delta x_i/2}^{x_i + \Delta x_i/2} \int_{y_j - \Delta y_j/2}^{y_j + \Delta y_j/2} \omega(x,y) \phi c_t \frac{\partial p(x,y)}{\partial t} - \omega(x,y)\, d(x,y) \left[\frac{\partial}{\partial x} \left(\frac{\bar{\bar{k}}}{\mu} \nabla p(x,y) \right) + \frac{\partial}{\partial y} \left(\frac{\bar{\bar{k}}}{\mu} \nabla_p \right) \right] dx\,dy \right\} = 0$$

$$\sum_{i=0}^{N_x-1}\sum_{j=0}^{N_y-1}\Bigg\{ \int_{x_i-\Delta x_i/2}^{x_i+\Delta x_i/2}\int_{y_j-\Delta y_j/2}^{y_j+\Delta y_j/2} \omega(\cancelto{1}{x,y})\,\phi\, c_t\, d(x,y)\,\frac{\partial p(x,y)}{\partial t}\,dx\,dy$$

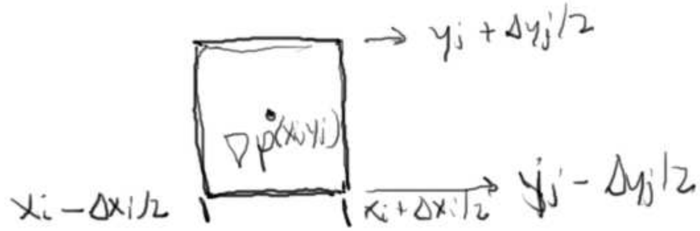
$$+\int_{x_i-\Delta x_i/2}^{x_i+\Delta x_i/2}\int_{y_j-\Delta y_j/2}^{y_j+\Delta y_j/2} \frac{\partial}{\partial x}\left(\omega(x,y)\,\cancelto{0}{d(x,y)}\right)\cdot\frac{\bar{\bar{k}}}{\mu}\nabla p\,dx\,dy$$

$$+\int_{x_i-\Delta x_i/2}^{x_i+\Delta x_i/2}\int_{y_j-\Delta y_j/2}^{y_j+\Delta y_j/2} \frac{\partial}{\partial y}\left(\omega(x,y)\,\cancelto{0}{d(x,y)}\right)\cdot\frac{\bar{\bar{k}}}{\mu}\nabla p\,dx\,dy$$

$$-\left[\int_{y_j-\Delta y_j/2}^{y_j+\Delta y_j/2}\omega(\cancelto{1}{x,y})\,d(x,y)\,\frac{\bar{\bar{k}}}{\mu}\nabla p\,dy\right]_{x_i-\Delta x_i/2}^{x_i+\Delta x_i/2}-\left[\int_{x_i-\Delta x_i/2}^{x_i+\Delta x_i/2}\omega(\cancelto{1}{x,y})\,d(x,y)\,\frac{\bar{\bar{k}}}{\mu}\nabla p\,dx\right]_{y_j-\Delta y_j/2}^{y_j+\Delta y_j/2}\Bigg\}=0$$

Let $\omega(x,y) = 1$ + use midpoint quadrature on intergals

Use FD for ∇p



$$\nabla p = \begin{Bmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \end{Bmatrix}$$

$$
\sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} \Bigg\{ \frac{\phi\, c_t\, d_{i,j}}{B_\alpha} \frac{\partial p_{i,j}}{\partial t} \Delta x_i \Delta y_j - d_{i+\frac{1}{2}}\, y_j\, \Delta y_j \frac{k_{i+\frac{1}{2},j}}{B_\alpha \mu} \left(\frac{p_{i+1,j} - p_{i,j}}{\Delta x_{i+\frac{1}{2}}} \right)
$$

$$
+ d_{i+\frac{1}{2}}\, y_j\, \Delta y_j \frac{k_{i-\frac{1}{2},j}}{B_\alpha \mu} \left(\frac{p_i - p_{i-1}}{\Delta x_{i+\frac{1}{2}}} \right) - d_{i,j+\frac{1}{2}}\, \Delta x_i \frac{k_{i,j+\frac{1}{2}}}{B_\alpha \mu} \left(\frac{p_{i,j+1} - p_{i,j}}{\Delta y_{j+\frac{1}{2}}} \right)
$$

$$
+ d_{i,j-\frac{1}{2}}\, \Delta x_i \frac{k_{i,j-\frac{1}{2}}}{B_\alpha \mu} \left(\frac{p_{i,j} - p_{i,j-1}}{\Delta y_{j-\frac{1}{2}}} \right) \Bigg\} = 0
$$

$$B_{i,j} = \frac{\phi_{i,j} \, c_t \, V_{i,j}}{B_\alpha} \quad \text{where } V_{i,j} = d_{i,j} \, \Delta x_i \, \Delta y_j$$

$$T_{i+1/2,j} = \frac{d_{i+1/2,j} \Delta y_j}{\Delta x_{i+1/2}} \; \frac{k_{i+1/2,j}}{B_\alpha \mu}$$

$$T_{i-1/2,j} = \frac{d_{i-1/2,j} \Delta y_j}{\Delta x_{i-1/2}} \; \frac{k_{i-1/2,j}}{B_\alpha \mu}$$

$$T_{i,j+1/2} = \frac{d_{i,j+1/2} \Delta x_i}{\Delta y_{i+1/2}} \; \frac{k_{i,j+1/2}}{B_\alpha \mu}$$

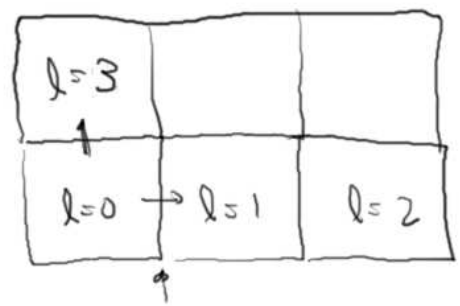
$$T_{i,j-1/2} = \frac{d_{i,j+1/2} \Delta x_i}{\Delta y_{i-1/2}} \; \frac{k_{i,j-1/2}}{B_\alpha \mu}$$

$$\sum_{i=0}^{n_x-1} \sum_{j=0}^{n_y-1} \left\{ B_{i,j} \frac{\partial p_{i,j}}{\partial t} + T_{i+1/2,j} \left(p_{i,j} - p_{i+1,j} \right) + T_{i-1/2,j} \left(p_{i,j} - p_{i-1,j} \right) + T_{i,j+1/2} \left(p_{i,j} - p_{i,j+1} \right) + T_{i,j-1/2} \left(p_{i,j} - p_{i,j-1} \right) \right\} = 0$$

$$B_{0,0}\frac{\partial p_{0,0}}{\partial t} + B_{1,0}\frac{\partial p_{1,0}}{\partial t} + B_{2,0}\frac{\partial p_{2,0}}{\partial t} + \dots + B_{N_x-1,0}\frac{\partial p_{N_x-1,0}}{\partial t} + B_{0,1}\frac{\partial p_{0,1}}{\partial t} + B_{1,1}\frac{\partial p_{1,1}}{\partial t}$$

$$+ B_{N_x-1,N_y-1}\frac{\partial p_{N_x-1,N_y-1}}{\partial t} + T_{1/2,0}\left(p_{0,0} - p_{1,0}\right) + \dots$$

Recall $\ell = j \cdot N_x + i$



$\underline{N_x = 3}$

$B_{0,0} \to B_{\ell=0}$ likewise $p_{0,0} \to p_{\ell=0}$

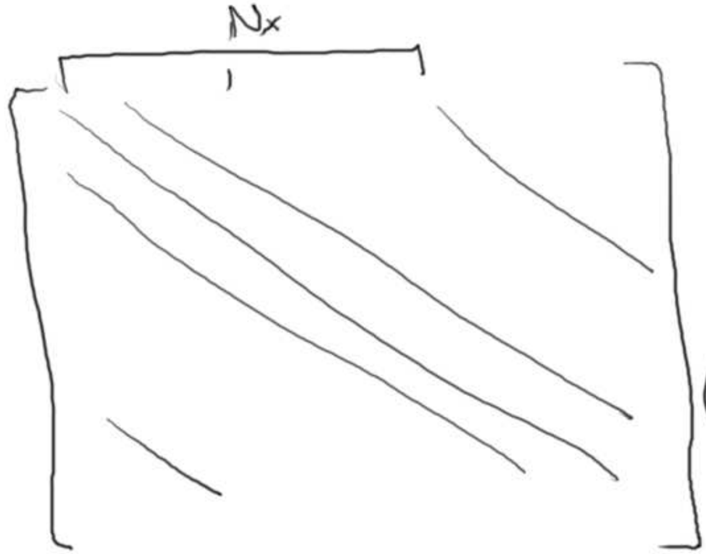
$B_{1,0} \to B_{\ell=1}$ " $p_{1,0} \to p_{\ell=1}$

$$B_{\ell=0}\frac{\partial p_{\ell=0}}{\partial t} + B_{\ell=1}\frac{\partial p_{\ell=1}}{\partial t} + \dots + T_{1/2,0}\left(p_{\ell=0} - p_{\ell=1}\right) + \dots + \underline{T_{0,1/2}}\left(p_{\ell=0} - p_{\ell=N_x}\right) + \dots$$

$$\begin{bmatrix} B_{\ell=0} & & & \\ & B_{\ell=1} & & \\ & & \ddots & \\ & & & B_{\ell=N_x \times N_y} \end{bmatrix} \left\{\frac{\partial \vec{p}}{\partial t}\right\} + [T]\left\{\vec{p}\right\} = \left\{\vec{Q}\right\}$$

$$[T] = \begin{array}{c} \overbrace{\hphantom{T_{-1/2,0}+T_{1/2,0}+T_{0,-1/2}+T_{0,1/2} \quad -T_{1/2,0} \quad 0 \quad 0}}^{N_x} \\ \begin{bmatrix} T_{-1/2,0}+T_{1/2,0}+T_{0,-1/2}+T_{0,1/2} & -T_{1/2,0} & 0 & 0 & -T_{0,1/2} & 0 \\ -T_{1/2,0} & T_{1/2,0}+T_{3/2,0}+T_{0,-1/2}+T_{0,1/2} & & & & -T_{0,1/2} \\ & & \ddots & & & \\ & & & & & \end{bmatrix} \end{array}$$

$[T] =$



Discretize in time "implicit" or "explicit"

$$\left([T] + \frac{1}{\Delta t}[B]\right)\vec{p}^{\,n+1} = \left(\frac{1}{\Delta t}[B]\{\vec{p}^{\,n}\}\right)$$

Exactly as in 1D