

$$\sum_{i=0}^{N-1} \left[ \frac{\partial p(x_i, t)}{\partial t} - \alpha \frac{\partial^2 p(x_i, t)}{\partial x^2} \right] \approx 0$$

$$\frac{\partial^2 p(x)}{\partial x^2} = \frac{p(x_i + \Delta x_i) - 2p(x_i) + p(x_i - \Delta x_i))}{\Delta x_i^2} = \frac{p(x_{i+1}) - 2p(x_i) + p(x_{i-1}))}{\Delta x^2}$$

The diagram shows a horizontal line representing a 1D lattice. Three points are marked with dots and labeled  $x_0$ ,  $x_1$ , and a third point to the right. Brackets below the line indicate distances:  $\Delta x_i$  between  $x_0$  and  $x_1$ ,  $\Delta x_i^2$  between  $x_0$  and the third point, and  $\Delta x$  between  $x_1$  and the third point. A curly brace on the right side of the line indicates the continuation of the lattice.

$$\approx \left[ \frac{p_{i+1} - 2p_i + p_{i-1}}{\Delta x^2} \right]$$

$$\frac{\partial p_0}{\partial t} + \frac{\alpha}{\Delta x^2} [-p_1 + 2p_0 - p_{-1}] + \frac{\partial p_1}{\partial t} + \frac{\alpha}{\Delta x^2} [-p_2 + 2p_1 - p_0]$$

$$+ \dots + \frac{\partial p_{N-2}}{\partial t} + \frac{\alpha}{\Delta x^2} [-p_{N-1} + 2p_{N-2} - p_{N-3}]$$

$$+ \frac{\partial p_{N-1}}{\partial t} + \frac{\alpha}{\Delta x^2} [-p_N + 2p_{N-1} - p_{N-2}] = 0$$

$2p_0 - p_{-1}$   
 $p_0(2 - \frac{p_{-1}}{p_0})$



$$\underbrace{\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & \dots & 0 & 1 \end{bmatrix}}_{[I]} \begin{Bmatrix} \frac{\partial p_0}{\partial t} \\ \frac{\partial p_1}{\partial t} \\ \vdots \\ \frac{\partial p_{N-1}}{\partial t} \end{Bmatrix} = \frac{\partial \vec{p}}{\partial t}$$

$$+ \underbrace{\begin{bmatrix} 2 - \frac{p_{-1}}{p_0} & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \vdots & & & \ddots & & \\ 0 & \dots & & -1 & 2 & -1 \\ & & & & & -1 \end{bmatrix}}_A \begin{Bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_{N-2} \\ p_{N-1} \end{Bmatrix} = \vec{0}$$

$\vec{p}$

$$[I] \left\{ \frac{\partial \vec{p}}{\partial t} \right\} + \frac{\alpha}{\Delta x^2} [A] \left\{ \vec{p} \right\} = \left\{ \vec{0} \right\} \leftarrow$$

$$\left\{ \frac{\partial \vec{p}}{\partial t} \right\} = -\frac{\alpha}{\Delta x^2} [I]^{-1} [A] \left\{ \vec{p} \right\}$$

$$\underbrace{\left\{ \frac{\partial \vec{p}}{\partial t} \right\} = -\frac{\alpha}{\Delta x^2} [A] \left\{ \vec{p} \right\}}_{\text{ODE}} \Rightarrow \underbrace{\left\{ \vec{p} \right\} = \exp\left(-\frac{\alpha}{\Delta x^2} [A]\right) \left\{ \vec{p}_0 \right\}}_{\text{solution}}$$

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Forward-diff in  $t$

$$\frac{\partial \vec{p}}{\partial t} = \frac{\vec{p}(t^n + \Delta t) - \vec{p}(t^n)}{\Delta t} = \frac{\vec{p}^{n+1} - \vec{p}^n}{\Delta t}$$