PGE 383 Advanced Geomechanics Midterm Exam

March 27, 2025

Instructions: You are allowed any printed resource (e.g. books, notes, etc.) to complete the exam except consultation with another student or a computer (including cell phones).

Problem 1

Valid constitutive models must obey the *Principle of Material Reference Frame Indifference*, i.e. they should be invarient to rigid rotations. Mathematically, for a Cauchy stress σ and a orthoganal time-dependent rotation tensor $\mathbf{R} = \mathbf{R}(t)$ we might say that the stress is invariant to a rotation if

$$\sigma = \mathbf{R}\sigma\mathbf{R}^\top. \tag{1}$$

However, often in plasticity modeling, we use rate-forms of constitutive models. Is the rate-of-Cauchy stress material reference frame indifferent? Explain.

Solution

Take the time derivative of both sides of the equation, i.e.

$$\dot{\sigma} = \dot{\mathbf{R}}\sigma\mathbf{R}^\top + \mathbf{R}\dot{\sigma}\mathbf{R}^\top + \mathbf{R}\sigma\dot{\mathbf{R}}^\top$$

This clearly is not equivalent to 1, therefore the rate-of-Cauchy stress **is not** material reference frame indifferent.

Problem 2

For the stress tensor

$$\sigma = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 3 & 6 \\ 4 & 6 & 0 \end{bmatrix}$$

and it's inverse

$$\sigma^{-1} = \begin{bmatrix} \frac{3}{10} & -\frac{1}{5} & \frac{1}{10} \\ -\frac{1}{5} & \frac{2}{15} & \frac{1}{10} \\ \frac{1}{10} & \frac{1}{10} & -\frac{1}{20} \end{bmatrix}$$

find a direction $\hat{\mathbf{n}}$, such that the traction vector on a plane normal to $\hat{\mathbf{n}}$ has components $t_1 = t_2 = 0$, and determine t_3 on that plane.

Solution

First we will solve for the unknown normal vector using the Cauchy stress equation, i.e. $\vec{t} = \sigma^T \hat{n}$. Where $\vec{t}^T = [0\ 0\ 1]$, we can use 1 here in place of t_3 because any arbrary t_3 that appears in the solution, will cancel when dividing the solution by its norm to create the normal vector.

Solving for $\hat{\mathbf{n}}$

$$\sigma^{-\top}\vec{\mathbf{t}} = \hat{\mathbf{n}}$$

$$\begin{bmatrix} \frac{3}{10} & -\frac{1}{5} & \frac{1}{10} \\ -\frac{1}{5} & \frac{2}{15} & \frac{1}{10} \\ \frac{1}{10} & \frac{1}{10} & -\frac{1}{20} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{10} \\ \frac{1}{10} \\ -\frac{1}{20} \end{bmatrix}$$

Ensure $\hat{\mathbf{n}}$ is a unit vector, i.e.

$$\hat{\mathbf{n}}^T = \left[\frac{1}{10}, \frac{1}{10}, -\frac{1}{20}\right] / (3/5) = \left[\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right]$$

Now can use this normal vector to determine t_3.

$$\sigma^\top \hat{\mathbf{n}} = \vec{\mathbf{t}}$$

$$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 3 & 6 \\ 4 & 6 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{20}{3} \end{bmatrix}$$

So the final answers are

$$\hat{\mathbf{n}}^T = \left[\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right]$$

and

$$t_3 = \frac{20}{3}$$

Problem 3

Under what condition are these two statements equavalent

$$\det(\mathbf{F}) = 1$$

$$\frac{\partial v_i}{\partial x_i} = 0$$

Explain your answer.

Solution

The correct answer to both conditions would be the *incompressibility* condition, from conservation of mass

$$\det(\mathbf{F}) = 1 = \frac{\rho_o}{\rho}$$

which implies that density doesn't change or the motion is *volume preserving* when viewed from the reference configuration. Likewise, in Eulerian form, conservation of mass is

$$\frac{D\rho}{Dt} + \rho\frac{\partial v_i}{\partial x_i} = 0$$

When viewed from the position of a moving particle, incompressibility implies $\frac{D\rho}{Dt} = 0$ therefore

$$\frac{\partial v_i}{\partial x_i} = 0$$

Problem 4

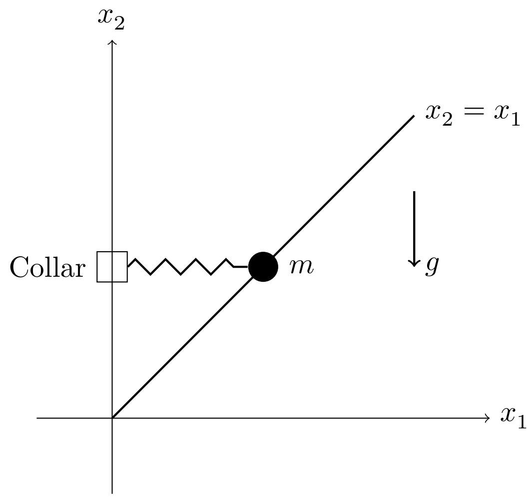
If a ductile material undergoing uniaxial stress, exhibits linear isotropic hardening, with hardening modulus H and yield stress Y, write a von Mises yield function for this material as a function of these material constants and equivalent plastic strain ε^p.

Solution

$$f = \sqrt{3J_2} - Y - H\varepsilon^p$$

Problem 5

Consider the illustration below



A mass m is constrainted to move on the line $x_2 = x_1$ under the influence of gravity g. A spring with spring constant k is affixed horizontally between the mass and a massless, frictionless collar that moves along the x_2 axis. The kinetic and potential energy for this system are

$$T = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2\right), \qquad U = \frac{1}{2}kx_1^2 + mgx_2 \tag{2}$$

Use Hamilton's extended principle to derive the equation of motion for this system. Express your final answer in terms of x_1 and its derivatives only.

From the course notes,

$$\int_{t_1}^{t_2} \delta T \mathrm{d}t = -\int_{t_1}^{t_2} m(\ddot{x}_1\delta x_1 + \ddot{x}_2\delta x_2)\mathrm{d}t,$$

now computing the variation of U

$$\delta U = kx_1\delta x_1 + mg\delta x_2,$$

and the variation of the constraint equation $C = \lambda(x_2 - x_1) = 0$,

$$\delta C = \lambda(\delta x_2 - \delta x_1).$$

Extended Hamilton's principle then states that amoung admissible motions, the actual motion of the body is

$$\int_{t_1}^{t_2} \left(\delta(T - U) + \delta C\right)\mathrm{d}t = 0$$

Substituting the expressions above and invoking the Fundamendal Lemma of the Calculus of Variations gives the following equations

$$\begin{aligned} \delta x_1 : \quad & 0 = -m\ddot{x}_1 - kx_1 - \lambda \\ \delta x_2 : \quad & 0 = -m\ddot{x}_2 + mg + \lambda \\ \delta\lambda : \quad & 0 = x_2 - x_1 \end{aligned}$$

Adding the first two equations together to eliminate λ, and using the final equation to note that $\ddot{x}_2 = \ddot{x}_1$ gives

$$0 = 2m\ddot{x}_1 + kx_1 - mg$$

or

$$\ddot{x}_1 = \frac{mg - kx_1}{2m}$$