assignment1_solution

March 11, 2025

1 Homework Assignment 1

Note: This is a computable IPython notebook who's source code can be downloaded here.

1.1 Problem 1

The motion of a certain continuous medium is defined by the equations

$$x_1 = \frac{1}{2}\left(X_1 + X_2\right)e^t + \frac{1}{2}\left(X_1 - X_2\right)e^{-t},$$
$$x_2 = \frac{1}{2}\left(X_1 + X_2\right)e^t - \frac{1}{2}\left(X_1 - X_2\right)e^{-t},$$
$$x_3 = X_3$$

1. Compute the following
   1. The Green-Lagrange strain tensor E
   2. The linear (small) strain tensor ε

   Plot the 11, 22, and 12 components of E and ε on the same figure from time $t = 0$ to $t = 0.05$.
2. Compute the following
   1. The rate-of-deformation tensor D
   2. The rate-of-change of the small strain tensor $\dot{\varepsilon} = \frac{d\varepsilon}{dt}$

   Plot the 11, 22, and 12 components of D and $\dot{\varepsilon}$ on the same figure from time $t = 0$ to $t = 0.05$.

Solution

This cell loads some important packages from sympy, numpy, and matplotlib that I will use the perform calculations and display the results neatly. In order to run this notebook, you will have to have these packages installed in your Python distribution.

[1]:
```
from sympy import *
from sympy.matrices import *
import sympy.mpmath
from sympy.utilities.lambdify import lambdify
init_printing()

import numpy
```

```
%matplotlib inline
import matplotlib.pyplot as plt
#plt.style.available
#plt.style.use('bmh')
```

Defining which variables will be "symbolic" in nature, i.e., they will not take on numerical values.

[2]:
```
t, X1, X2, X3 = symbols('t, X_1, X_2, X_3')
```

This defines the deformation mapping as in the problem statement.

[3]:
```
x1 = Rational(1, 2) * (X1 + X2) * exp(t) + Rational(1, 2) * (X1 - X2) * exp(-t)
x2 = Rational(1, 2) * (X1 + X2) * exp(t) - Rational(1, 2) * (X1 - X2) * exp(-t)
x3 = X3
```

Now we compute the deformation gradient.

[4]:
```
F = simplify(Matrix([[diff(x1, X1), diff(x1, X2), diff(x1, X3)],
            [diff(x2, X1), diff(x2, X2), diff(x2, X3)],
            [diff(x3, X1), diff(x3, X2), diff(x3, X3)]])); F
```

[4]:

$$\begin{bmatrix} \cosh(t) & \sinh(t) & 0 \\ \sinh(t) & \cosh(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And the Green-Lagrange strain

[5]:
```
E = Rational(1, 2) * (F.T * F - eye(3)); simplify(E)
```

[5]:

$$\begin{bmatrix} \sinh^2(t) & \frac{1}{2}\sinh(2t) & 0 \\ \frac{1}{2}\sinh(2t) & \sinh^2(t) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We can immediatelly evaluate the linear "small" strain as well, directly with the deformation gradient.

[6]:
```
epsilon = simplify(Rational(1,2) * (F.T + F) - eye(3)); simplify(epsilon)
```

[6]:

$$\begin{bmatrix} \cosh(t) - 1 & \sinh(t) & 0 \\ \sinh(t) & \cosh(t) - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This turns our symbolic output into actual functions of t that we can evaluate and plot

[7]:
```
E11_function = lambdify(t, E[0,0], "numpy")
epsilon11_function = lambdify(t, epsilon[0,0], "numpy")

E22_function = lambdify(t, E[1,1], "numpy")
```

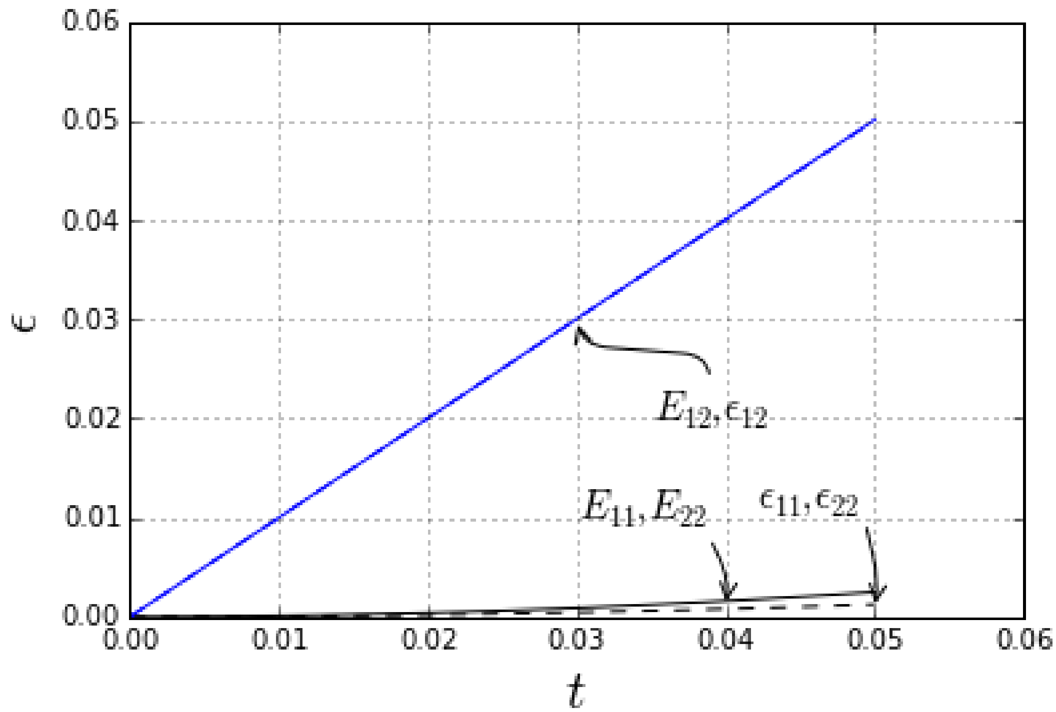
```
epsilon22_function = lambdify(t, epsilon[1,1], "numpy")

E12_function = lambdify(t, E[0,1], "numpy")
epsilon12_function = lambdify(t, epsilon[0,1], "numpy")
```

Evaluating the functions

[9]:
```
t0 = numpy.linspace(0.0,0.05)

E11 = E11_function(t0)
epsilon11 = epsilon11_function(t0)

E22 = E22_function(t0)
epsilon22 = epsilon22_function(t0)

E12 = E12_function(t0)
epsilon12 = epsilon12_function(t0)
```

Plotting the results

[10]:
```
fig = plt.figure(1)
ax = fig.add_subplot(111,xlabel='$t$', ylabel='$\epsilon$')
ax.plot(t0, E22,'k-', t0, epsilon22, 'k--',
         t0, E12,'b-', t0, epsilon12, 'b--');
plt.grid()
ax.annotate('$E_{11}, E_{22}$', xy=(0.04, 0.001),  xycoords='data',
            xytext=(0.03, 0.01), textcoords='data',
            arrowprops=dict(arrowstyle="->",␣
 ↪connectionstyle="arc,angleA=0,armA=20,angleB=80,armB=10,rad=10"),
            fontsize='15'
           )
ax.annotate('$\epsilon_{11}, \epsilon_{22}$', xy=(0.05, 0.0008), ␣
 ↪xycoords='data',
            xytext=(0.042, 0.011), textcoords='data',
            arrowprops=dict(arrowstyle="->",␣
 ↪connectionstyle="arc,angleA=0,armA=20,angleB=80,armB=10,rad=10"),
            fontsize='15'
           )
ax.annotate('$E_{12}, \epsilon_{12}$', xy=(0.03, 0.03),  xycoords='data',
            xytext=(0.035, 0.02), textcoords='data',
            arrowprops=dict(arrowstyle="->",␣
 ↪connectionstyle="arc,angleA=90,armA=20,angleB=-80,armB=10,rad=10"),
            fontsize='15'
           );
ax.xaxis.label.set_size(20)
ax.yaxis.label.set_size(20)
```



Now compute the rate-of-deformation tensor, i.e. the symmetric part of the velocity gradient

[11]:

```
Fdot = F.diff(t);

L = expand(Fdot * F.inv());

D = simplify(Rational(1,2) * (L.T + L)); D
```

[11]:

$$\begin{bmatrix}0 & 1 & 0\\1 & 0 & 0\\0 & 0 & 0\end{bmatrix}$$

And the linear "small" strain-rate, $\dot{\epsilon}$

[12]:

```
epsilon_dot = epsilon.diff(t); epsilon_dot
```

[12]:

$$\begin{bmatrix}\sinh{\left(t \right)} & \cosh{\left(t \right)} & 0\\\cosh{\left(t \right)} & \sinh{\left(t \right)} & 0\\0 & 0 & 0\end{bmatrix}$$

Now we will turn the symbolic small strain components into functions that we can evaluate in time. It's not necassary to perform this operation on the rate-of-deformation tensor because it has constant components

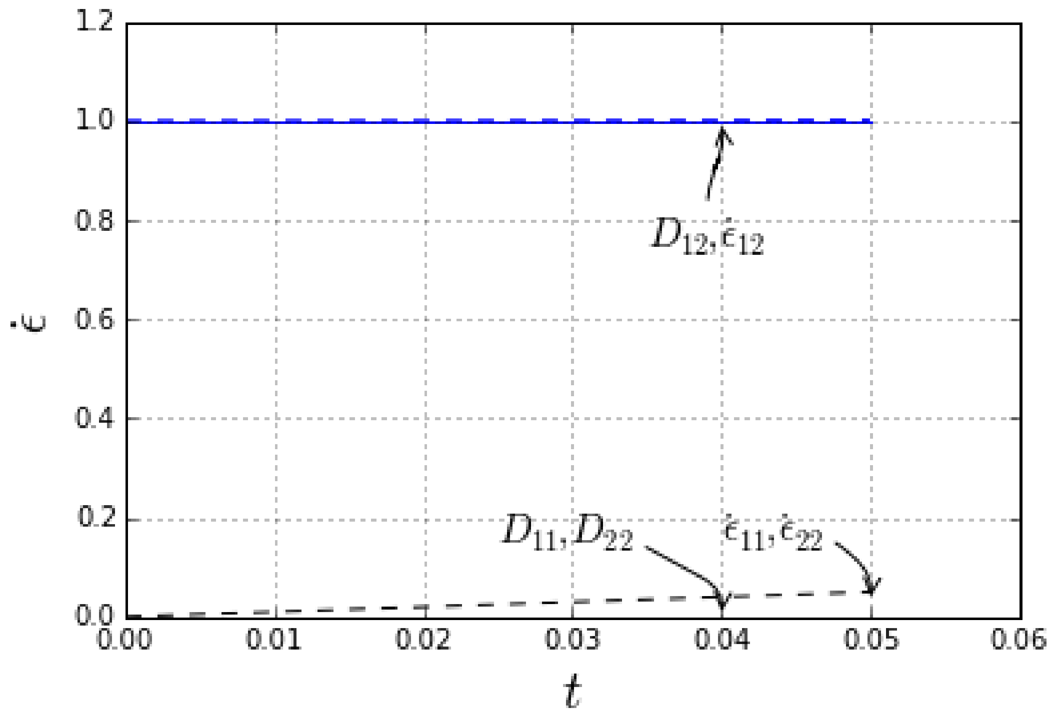
[13]:
```
epsilon11_dot_function = lambdify(t, epsilon_dot[0,0], "numpy")

epsilon22_dot_function = lambdify(t, epsilon_dot[1,1], "numpy")

epsilon12_dot_function = lambdify(t, epsilon_dot[0,1], "numpy")
```

Evaluating the functions

[14]:
```
D11 = numpy.zeros_like(t0)
epsilon11_dot = epsilon11_dot_function(t0)

D22 = D11
epsilon22_dot = epsilon22_dot_function(t0)

D12 = numpy.ones_like(t0)
epsilon12_dot = epsilon12_dot_function(t0)
```

Plotting the results

[15]:
```
fig = plt.figure(2)
ax = fig.add_subplot(111,xlabel='$t$', ylabel='$\dot{\epsilon}$')
ax.plot(t0, D11,'k-', t0, epsilon11_dot, 'k--',
         t0, D12,'b-', t0, epsilon12_dot, 'b--');
plt.grid()
ax.annotate('$D_{11}, D_{22}$', xy=(0.04, 0.0),  xycoords='data',
            xytext=(0.025, 0.15), textcoords='data',
            arrowprops=dict(arrowstyle="->",␣
 ↪connectionstyle="arc,angleA=0,armA=20,angleB=80,armB=10,rad=10"),
            fontsize='15'
           )
ax.annotate('$\dot{\epsilon}_{11}, \dot{\epsilon}_{22}$', xy=(0.05, 0.03), ␣
 ↪xycoords='data',
            xytext=(0.04, 0.15), textcoords='data',
            arrowprops=dict(arrowstyle="->",␣
 ↪connectionstyle="arc,angleA=0,armA=20,angleB=80,armB=10,rad=10"),
            fontsize='15'
           )
ax.annotate('$D_{12}, \dot{\epsilon}_{12}$', xy=(0.04, 1.0),  xycoords='data',
            xytext=(0.035, 0.75), textcoords='data',
            arrowprops=dict(arrowstyle="->",␣
 ↪connectionstyle="arc,angleA=90,armA=20,angleB=-80,armB=10,rad=10"),
            fontsize='15'
           );
ax.xaxis.label.set_size(20)
ax.yaxis.label.set_size(20)
```



1.2 Problem 2

Given the following stress tensor

$$\sigma = \begin{bmatrix} 36 & 27 & 0 \\ 27 & -36 & 0 \\ 0 & 0 & 18 \end{bmatrix}$$

Find:

1. the components of the traction vector acting on a plane with unit normal vector $\hat{n}^T = [2/3, -2/3, 1/3]$
2. the magnitude of the traction vector found in (a)
3. its component in the direction of the normal
4. a. the angle between the traction vector and the normal

Solution

Defining the stress tensor and normal vector

[16]:
```
sigma = Matrix([[36, 27, 0],[27, -36, 0],[0, 0, 18]])
n = Matrix([Rational(2, 3), Rational(-2, 3), Rational(1, 3)])
```

The traction vector is then $\vec{t} = \sigma^T \hat{n}$ according to the Cauchy stress equation.

[17]:
```
t = sigma.T * n; t
```
[17]:

$$\begin{bmatrix} 6 \\ 42 \\ 6 \end{bmatrix}$$

Computing the magnitude

[18]:
```
sympy.mpmath.mp.pretty = True
magnitude = sympy.mpmath.norm(t,2); magnitude
```
[18]: 42.8485705712571

And the projection in the direction of the normal.

[19]:
```
t.T * n
```
[19]:

$$\begin{bmatrix} -22 \end{bmatrix}$$

The angle between the normal and the traction vector (in radians)

[20]:
```
acos(((t / magnitude).T * n)[0])
```
[20]:

$$2.10998044394962$$

or in degree

[21]:
```
_ * 180. / numpy.pi
```
[21]:

$$120.892974293453$$

1.3 Problem 3

Given the following stress tensor

$$\sigma = \begin{bmatrix} 18 & 0 & 24 \\ 0 & -50 & 0 \\ 24 & 0 & 32 \end{bmatrix}$$

Find:

1. the principle stresses $\sigma_I, \sigma_{II}, \sigma_{III}$
2. the three invariants I_1, I_2, I_3
3. the deviatoric stress
4. the two nonzero invariants of the deviatoric stress, i.e. J_2, J_3

Solution

Defining the stress tensor

[22]:
```
sigma = Matrix([[18, 0, 24],[0, -50, 0],[24, 0, 32]])
```

Here we use sympy to diagonalize (or find the eigenvalues, they are shown on the diagonal of the matrix. We then define $\sigma_I > \sigma_{II} > \sigma_{III}$ accordingly.

[23]:
```
_, D = sigma.diagonalize();
sigma1 = D[2,2];sigma2 = D[1,1]; sigma3 = D[0,0]; D
```

[23]:

$$\begin{bmatrix} -50 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 50 \end{bmatrix}$$

The first, second, and third invariants

[24]:
```
I1 = sigma1 + sigma2 + sigma3; I1
```

[24]:

$$0$$

[25]:
```
I2 = sigma1 * sigma2 + sigma1 * sigma3 + sigma2 * sigma3; I2
```

[25]:

$$-2500$$

[26]:
```
I3 = sigma1 * sigma2 * sigma3; I3
```

[26]:

$$0$$

The deviatoric stress

[27]:
```
Sij = sigma - 1. / 3. * I1 * eye(3); Sij
```

[27]:

$$\begin{bmatrix} 18 & 0 & 24 \\ 0 & -50 & 0 \\ 24 & 0 & 32 \end{bmatrix}$$

Here we perform the same procedure on the deviatoric stress and compute the invariants.

[28]:
```
_, D = Sij.diagonalize();
Sij1 = D[2,2]; Sij2 = D[1,1]; Sij3 = D[0,0];

J2 = Sij1 * Sij2 + Sij1 * Sij3 + Sij2 * Sij3; J2
```

[28]:

$$-2500$$

[29]:
```
J3 = Sij1 * Sij2 * Sij3; J3
```

[29]:

$$0$$

1.4 Problem 4

Show that

$$\frac{\partial J_2}{\partial \sigma_{ij}} = S_{ij}$$

where J_2 is the second invariant of the deviatoric stress tensor, S_{ij}.

Solution

$$\begin{aligned}
\frac{\partial J_2}{\partial \sigma_{ij}} &= \frac{\partial}{\partial \sigma_{ij}} (J_2) && (1)\\
&= \frac{\partial}{\partial \sigma_{ij}} \left(\frac{1}{2} S_{kl} S_{kl}\right) && (2)\\
&= \frac{1}{2}\left(\frac{\partial}{\partial \sigma_{ij}} (S_{kl}) S_{kl} + S_{kl} \frac{\partial}{\partial \sigma_{ij}} (S_{kl})\right) && (3)\\
&= S_{kl} \frac{\partial}{\partial \sigma_{ij}} (S_{kl}) && (4)\\
&= S_{kl} \frac{\partial}{\partial \sigma_{ij}} \left(\sigma_{kl} - \frac{1}{3}\sigma_{mm}\delta_{kl}\right) && (5)\\
&= S_{kl} \left(\delta_{il}\delta_{kj} - \frac{1}{3}\delta_{im}\delta_{jm}\delta_{kl}\right) && (6)\\
&= S_{kl} \left(\delta_{il}\delta_{kj} - \frac{1}{3}\delta_{ij}\delta_{kl}\right) && (7)\\
&= S_{ij} - \frac{1}{3} S_{kk}\delta_{ij} && (8)\\
&= S_{ij} && (9)
\end{aligned}$$

because

$$S_{kk} = 0$$

by definition of S being a deviatoric tensor.

1.5 Problem 5

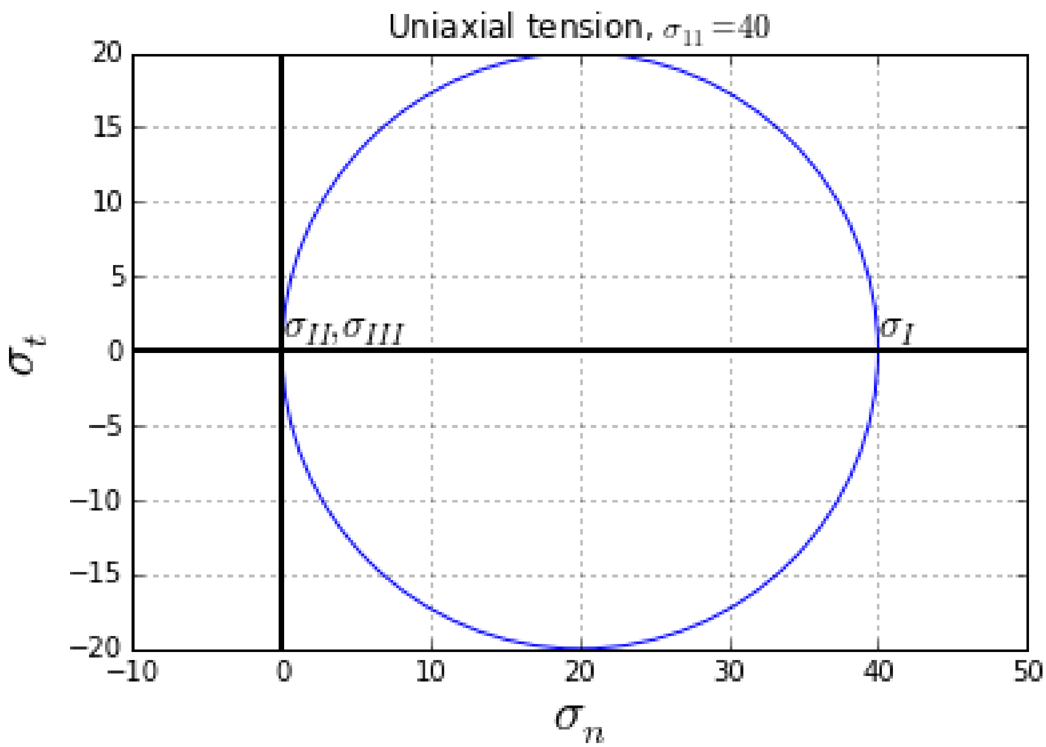
For each of the following stress states (values not given are zero), plot the three Mohr's circles and determine the maximum shear stress.

1. Uniaxial tension $\sigma_{11} = 40$
2. Biaxial stress $\sigma_{11} = -10, \sigma_{22} = 30$

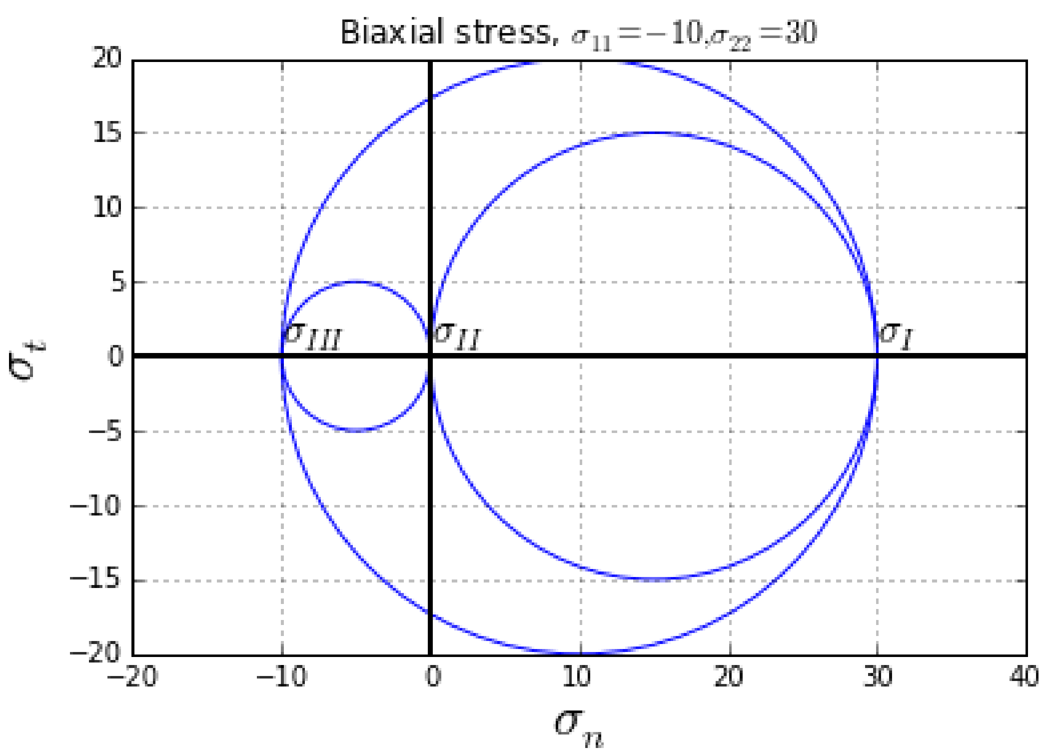
3. Hydrostatic tension of magnitude 100 psi
4. $\sigma_{11} = -60, \sigma_{22} = 100, \sigma_{33} = 40$
5. $\sigma_{11} = 10, \sigma_{22} = 40, \sigma_{21} = \sigma_{12} = 20$

Solution

[30]:
```
fig = plt.figure()
ax = fig.add_subplot(111,xlabel='$\sigma_n$',␣
 ↪ylabel='$\sigma_t$',title='Uniaxial tension, $\sigma_{11}=40$')
ax.xaxis.label.set_size(20)
ax.yaxis.label.set_size(20)
ax.axhline(0, color='black', lw=2)
ax.axvline(0, color='black', lw=2)
ax.annotate('$\sigma_{II},\sigma_{III}$', xy=(0.04, 1.0),  xycoords='data',␣
 ↪fontsize='15');
ax.annotate('$\sigma_{I}$', xy=(40, 1.0),  xycoords='data', fontsize='15');
circ = plt.Circle((0.5*(40-0), 0), radius=20, fill=False, color='b')
ax.add_patch(circ)
plt.axis('equal')
plt.grid()
plt.show()
```



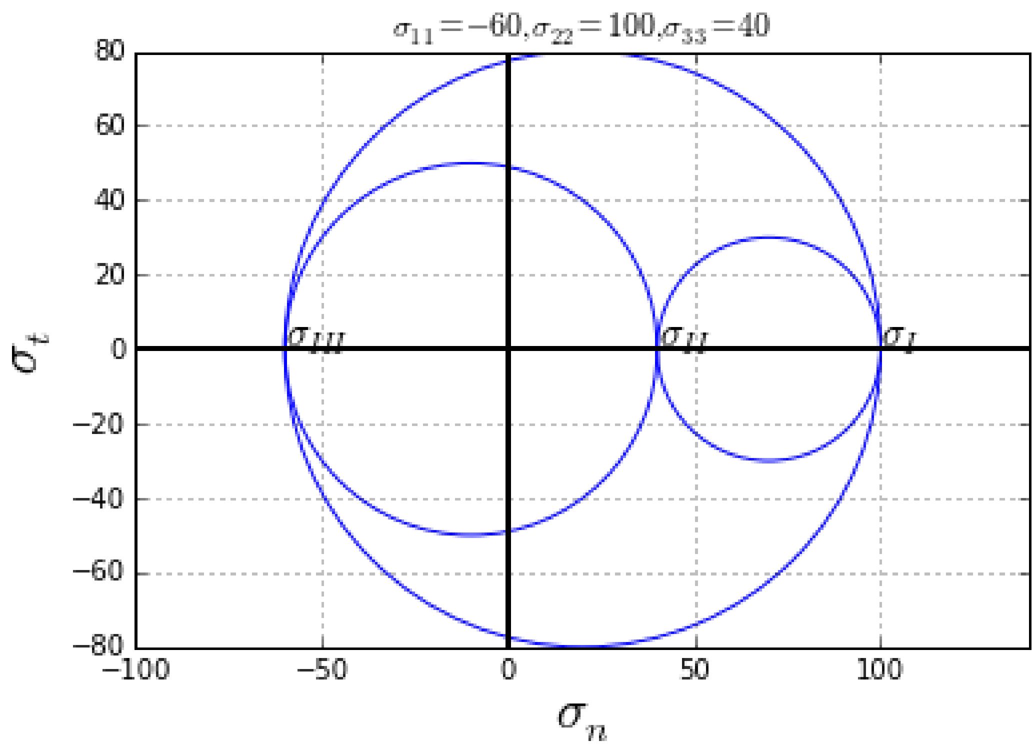
[31]:
```
fig = plt.figure()
ax = fig.add_subplot(111,xlabel='$\sigma_n$',␣
 ↪ylabel='$\sigma_t$',title='Biaxial stress, $\sigma_{11}=-10,␣
 ↪\sigma_{22}=30$')
ax.xaxis.label.set_size(20)
ax.yaxis.label.set_size(20)
ax.axhline(0, color='black', lw=2)
ax.axvline(0, color='black', lw=2)
ax.annotate('$\sigma_{III}$', xy=(-10, 1.0),  xycoords='data', fontsize='15');
ax.annotate('$\sigma_{II}$', xy=(0.04, 1.0),  xycoords='data', fontsize='15');
ax.annotate('$\sigma_{I}$', xy=(30, 1.0),  xycoords='data', fontsize='15');
circ1 = plt.Circle((0.5*(30+(-10)), 0), radius=20, fill=False, color='b')
circ2 = plt.Circle((0.5*(30+(0)), 0), radius=15, fill=False, color='b')
circ3 = plt.Circle((-5, 0), radius=5, fill=False, color='b')
ax.add_patch(circ1)
ax.add_patch(circ2)
ax.add_patch(circ3)
plt.axis('equal')
plt.grid()
plt.show()
```



For the hydostatic tension case, there are no Mohr's circles to draw because $\sigma_I = \sigma_{II} = \sigma_{III}$

[32]:

```
fig = plt.figure()
ax = fig.add_subplot(111,xlabel='$\sigma_n$', ylabel='$\sigma_t$',
                     title='$\sigma_{11} = -60, \sigma_{22} = 100, \sigma_{33}␣
 ↪= 40$')
ax.xaxis.label.set_size(20)
ax.yaxis.label.set_size(20)
ax.axhline(0, color='black', lw=2)
ax.axvline(0, color='black', lw=2)
ax.annotate('$\sigma_{III}$', xy=(-60, 1.0),  xycoords='data', fontsize='15');
ax.annotate('$\sigma_{II}$', xy=(40, 1.0),  xycoords='data', fontsize='15');
ax.annotate('$\sigma_{I}$', xy=(100, 1.0),  xycoords='data', fontsize='15');
circ1 = plt.Circle((0.5*(100+(-60)), 0), radius=80, fill=False, color='b')
circ2 = plt.Circle((70, 0), radius=30, fill=False, color='b')
circ3 = plt.Circle((-10, 0), radius=50, fill=False, color='b')
ax.add_patch(circ1)
ax.add_patch(circ2)
ax.add_patch(circ3)
plt.axis('equal')
plt.grid()
plt.show()
```



[33]:
```
sigma = Matrix([[10,20,0],[20,40,0],[0,0,0]])
_, D = sigma.diagonalize(); D
```

[33]:

$$\begin{bmatrix}0 & 0 & 0\\0 & 0 & 0\\0 & 0 & 50\end{bmatrix}$$

[34]:
```
fig = plt.figure()
ax = fig.add_subplot(111,xlabel='$\sigma_n$', ylabel='$\sigma_t$',
                     title='$\sigma_{11} = 10, \sigma_{22} = 40, \sigma_{21} =␣
 ↪\sigma_{12} = 20$')
ax.xaxis.label.set_size(20)
ax.yaxis.label.set_size(20)
ax.axhline(0, color='black', lw=2)
ax.axvline(0, color='black', lw=2)
ax.annotate('$\sigma_{II},\sigma_{III}$', xy=(0.04, 1.0),  xycoords='data',␣
 ↪fontsize='15');
ax.annotate('$\sigma_{I}$', xy=(50, 1.0),  xycoords='data', fontsize='15');
circ = plt.Circle((0.5*(50-0), 0), radius=25, fill=False, color='b')
ax.add_patch(circ)
plt.axis('equal')
plt.grid()
plt.show()
```