assignment2_solution

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1 Homework Assignment 2

1.1 Problem 1

1. Write out the conservation of linear momentum equations without using the summation convection, i.e. all components.

Solution

Writing out the components, using an x, y, z component convention

$$\rho \frac{Dv_x}{Dt} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho b_x$$
$$\rho \frac{Dv_y}{Dt} = \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \rho b_y$$
$$\rho \frac{Dv_z}{Dt} = \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho b_z$$

1.2 Problem 2

For most of the class, we've used a solid mechanics setting to motivate the physics of interest; however, the principles we've derived are general enough to apply to fluids as well. In an ideal nonviscous fluid there can be no shear stress. Hence, the stress tensor is entire hydrostatic, $\sigma_{ij} = -p\delta_{ij}$. Show that this leads to the following form of the momentum equation, known as Euler's equation of motion for a frictionless fluid:

$$-\frac{1}{\rho}\nabla p + \vec{b} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}$$

Solution

Starting with the conservation of momentum equation

$$
\begin{aligned}
\rho \frac{D\vec{v}}{Dt} &= \nabla \cdot \sigma + \rho \vec{b} \\
\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} &= \frac{1}{\rho} \nabla \cdot \sigma + \vec{b} \\
\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} &= -\frac{1}{\rho} \nabla \cdot (p\mathbf{I}) + \vec{b} \\
\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} &= -\frac{1}{\rho} \nabla p + \vec{b}
\end{aligned}
$$

1.3 Problem 3

Using the different forms of conservation of mass, derive an expression for $\frac{dJ}{dt}$ in terms of $\vec{v}$, where $J = \det(\mathbf{F})$

Solution

Using the material form of conservation of mass

$$
\begin{aligned}
\dot{J} &= \frac{\partial}{\partial t} \left(\det \left(\mathbf{F} \right) \right), \\
&= \frac{\partial \left(\det \left(\mathbf{F} \right) \right)}{\partial F_{ij}} \frac{\partial F_{ij}}{\partial t}, \\
&= \frac{\partial \left(\det \left(\mathbf{F} \right) \right)}{\partial F_{ij}} L_{ik} F_{kj}, \\
&= \det(\mathbf{F}) F_{ji}^{-1} L_{ik} F_{kj}, \\
&= \det(\mathbf{F}) L_{ik} \delta_{ki}, \\
&= \det(\mathbf{F}) L_{kk}, \\
&= J \frac{\partial v_k}{\partial x_k}, \\
&= J(\nabla \cdot \mathbf{v}),
\end{aligned}
$$

1.4 Problem 4

Assume that the internal-energy density can be given as $u = u(\epsilon, T)$, that the heat flux is governed by Fourier's law $\vec{q} = -k(T)\nabla T$, and that $r = 0$. Defining the specific heat $C = \frac{\partial u}{\partial T}$, write the equation resulting from combining these assumptions with the energy- balance equation.

Solution

Starting with the energy equation, for small displacements $D_{ij} \approx \dot{\varepsilon}_{ij}$

$$
\begin{aligned}
\rho\frac{Du}{Dt} &= \sigma : \dot{\varepsilon} + \nabla \cdot \vec{q} \\
\rho\frac{Du}{DT}\frac{DT}{Dt} + \rho\frac{Du}{D\epsilon}\frac{D\epsilon}{Dt} &= \sigma : \dot{\varepsilon} - \nabla \cdot (k(T)\nabla T) \\
\rho C\dot{T} + \rho\frac{Du}{D\epsilon}\dot{\epsilon} &= \sigma : \dot{\varepsilon} - \nabla k(T)\nabla T - k(T)\nabla^2 T \\
\rho C\dot{T} + \rho\frac{Du}{D\epsilon}\dot{\epsilon} &= \sigma : \dot{\varepsilon} - k(T)\nabla^2 T
\end{aligned}
$$

where it is understood that ϵ is a scalar internal state variable.

1.5 Problem 5

Show that, in an isotropic linearly elastic solid, the principal stress and principal strain directions coincide.

Solution

There are several ways to demonstrate this, first, let's assume that there is an orthonormal tensor $\mathbf{V}$ that diagonalizes the *strain tensor* ε, i.e. it consists of eigenvectors of the strain tensor and produces a diagonal tensor, $\varepsilon^{'}$ with the principle strains on the diagonal.

$$
\varepsilon^{'} = \mathbf{V}^{-1}\varepsilon\mathbf{V}
$$

We'll use the eigenvectors from the strain tensor to transform the stress tensor, if the result is diagonal, we know they are also eigenvectors of the stress tensor, that is the principle directions of both tensors are in the same direction.

$$
\begin{aligned}
\sigma'_{ij} &= V_{ik}^{-1}\sigma_{kl}V_{lj} \\
&= V_{ik}^{-1}\left(2\mu\varepsilon_{kl} + \lambda\varepsilon_{mm}\delta_{kl}\right)V_{lj} \\
&= 2\mu V_{ik}^{-1}\varepsilon_{kl}V_{lj} + \lambda\varepsilon_{mm}V_{il}^{-1}V_{lj} \\
&= 2\mu V_{ik}^{-1}\varepsilon_{kl}V_{lj} + \lambda\varepsilon_{mm}\delta_{ij} \\
&= 2\mu\varepsilon'_{ij} + \lambda\varepsilon_{mm}\delta_{ij}
\end{aligned}
$$

By inspection we can see that both the first and second terms of the resulting expressions are diagonal by definition, therefore σ' is also diagonal and $\mathbf{V}$ also contains the eigenvectors of σ

1.6 Problem 6

Write the elastic modulus matrix C_{IJ} for an isotropic linearly elastic solid in terms of the Young's modulus E and the Poisson's ratio ν.

Solution

$$C_{IJ} = \frac{E}{2(1+\nu)} \begin{bmatrix} \frac{2(1-\nu)}{1-2\nu} & \frac{2\nu}{1-2\nu} & \frac{2\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{2\nu}{1-2\nu} & \frac{2(1-\nu)}{1-2\nu} & \frac{2\nu}{1-2\nu} & 0 & 0 & 0 \\ \frac{2\nu}{1-2\nu} & \frac{2\nu}{1-2\nu} & \frac{2(1-\nu)}{1-2\nu} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

1.7 Problem 7

Combine the generalized Hooke's law for an isotropic linearly elastic solid with the equations of motion and the definition of small strain in order to derive the equations of motion for such a solid entirely in terms of displacement, using

1. λ and μ
2. G and ν

Solution

$$\begin{aligned} \rho\frac{D^2\vec{u}}{Dt^2} &= \nabla\cdot\sigma + \rho\vec{b} \\ &= \nabla\cdot(2\mu\varepsilon + \lambda\mathrm{tr}(\varepsilon)\mathbf{I}) + \rho\vec{b} \\ &= \mu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda\frac{\partial^2 u_k}{\partial x_j \partial x_k}\delta_{ij} \\ &= \mu\frac{\partial^2 u_i}{\partial x_j^2} + \mu\frac{\partial^2 u_j}{\partial x_i \partial x_j} + \lambda\frac{\partial^2 u_k}{\partial x_i \partial x_k} \\ &= \mu\frac{\partial^2 u_i}{\partial x_j^2} + \mu\frac{\partial^2 u_j}{\partial x_i \partial x_j} + \lambda\frac{\partial^2 u_j}{\partial x_i \partial x_j} \\ &= \mu\frac{\partial^2 u_i}{\partial x_j^2} + (\mu+\lambda)\frac{\partial^2 u_j}{\partial x_i \partial x_j} \\ &= \mu\nabla^2\vec{u} + (\mu+\lambda)\nabla(\nabla\cdot\vec{u}) \\ &= G\nabla^2\vec{u} + \frac{G}{1-2\nu}\nabla(\nabla\cdot\vec{u}) \end{aligned}$$

1.8 Problem 8

1. Show that minimizing the integral

$$I = \int_{t_1}^{t_2} L(y, \dot{y}, t)\, dt$$

results in the Euler-Lagrange Equation, i.e.

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0$$

2. A mass m suspended from a spring with stiffness k has the following kinetic and potential energies,

$$T = \frac{1}{2}m\dot{y}^2 \quad U = \frac{1}{2}ky^2$$

Assume $L = T - U$ and use the Euler-Lagrange equation to derive the equation-of-motion for the spring-mass system.

Solution

1. Compute δI, i.e.

$$\delta I = \int_{t_1}^{t_2} \frac{\partial L}{\partial y}\delta y + \frac{\partial L}{\partial \dot{y}}\delta \dot{y}\mathrm{d}t = 0$$

integrate the second term by parts

$$\delta I = \int_{t_1}^{t_2} \frac{\partial L}{\partial y}\delta y - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right)\delta y\mathrm{d}t + \left[\frac{\partial L}{\partial \dot{y}}\delta y\right]_{t_1}^{t_2} = 0$$

the last term vanishes because the varations at t_1 and t_2 are fixed. Then using the fundamental lemma of the calculus of varations, we have

$$\frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0$$

2. The Lagrangian is

$$L = T - U = \frac{1}{2}m\dot{y}^2 - \frac{1}{2}ky^2.$$

Using the Euler-Lagrange equation:

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0,$$

we compute the necessary derivatives:

$$\frac{\partial L}{\partial y} = \frac{\partial}{\partial y}\left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}ky^2\right) = -ky,$$
$$\frac{\partial L}{\partial \dot{y}} = \frac{\partial}{\partial \dot{y}}\left(\frac{1}{2}m\dot{y}^2 - \frac{1}{2}ky^2\right) = m\dot{y}.$$

The time derivative of $\frac{\partial L}{\partial \dot{y}}$ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = \frac{d}{dt}(m\dot{y}) = m\ddot{y}.$$

Substituting into the Euler-Lagrange equation

$$-ky - m\ddot{y} = 0,$$

or, after rearranging

$$m\ddot{y} + ky = 0.$$