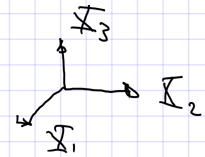
In indicial notation, repeated indices in "terms" imply summation

$$q_i - x_i = \delta_{ij}\left(Q_j - X_j\right) + \frac{\partial u_i}{\partial X_j}\left(Q_j - X_j\right) + H.O.T.'s$$

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{otherwise} \end{cases} \qquad a_i = \delta_{ij}\, a_j$$

$$\underbrace{q_i - x_i}_{dx_i} = \left(\delta_{ij} + \frac{\partial u_i}{\partial X_j}\right)\underbrace{\left(Q_j - X_j\right)}_{dX_j} + \mathcal{O}\left(\|\underbrace{\cancel{\vec{Q} - \vec{X}}}_{d\vec{X}}\|\right)^{\partial}$$



$$dx_i = \left(\delta_{ij} + \frac{\partial u_i}{\partial X_j}\right) dX_j$$

$$d\vec{x} = \underbrace{\left(I + (\nabla_X \vec{u})^T\right)}_{F} d\vec{X}$$

$F \rightarrow$ deformation gradient

$$\vec{X} = X_1 \hat{i} + X_2 \hat{j} + X_3 \hat{k} = X_1 \hat{e}_1 + X_2 \hat{e}_2 + X_3 \hat{e}_3 = \cancel{\sum_{i=1}^{3}} X_i \hat{e}_i = X_i \hat{e}_i$$

$$d\vec{x} = F\, d\vec{X}$$

$i=1$

$$dx_1 = \sum_j \delta_{1j}\, dX_j + \sum_j \frac{\partial u_1}{\partial X_j} dX_j$$
$$= \cancel{\delta_{11}}^{1} dX_1 + \quad ''$$

$$\begin{cases} dx_1 = dX_1 + \frac{\partial u_1}{\partial X_1} dX_1 + \frac{\partial u_1}{\partial X_2} dX_2 + \frac{\partial u_1}{\partial X_3} dX_3 \\ dx_2 = dX_2 + \frac{\partial u_2}{\partial X_1} dX_1 + \frac{\partial u_2}{\partial X_2} dX_2 + \frac{\partial u_2}{\partial X_3} dX_3 \\ dx_3 = dX_3 + \frac{\partial u_3}{\partial X_1} dX_1 + \frac{\partial u_3}{\partial X_2} dX_2 + \frac{\partial u_3}{\partial X_3} dX_3 \end{cases} \Rightarrow \begin{Bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{Bmatrix} = \begin{Bmatrix} dX_1 \\ dX_2 \\ dX_3 \end{Bmatrix} + \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} \\ \frac{\partial u_2}{\partial X_1} & \ddots \\ \frac{\partial u_3}{\partial X_1} & \cdots \end{bmatrix}$$

$$\vec{u} = u_1 \hat{e}_1 + u_2 \hat{e}_2 + u_3 \hat{e}_3$$

$$\nabla_X(\cdot) = \hat{e}_1 \frac{\partial(\cdot)}{\partial X_1} + \hat{e}_2 \frac{\partial(\cdot)}{\partial X_2} + \hat{e}_3 \frac{\partial(\cdot)}{\partial X_3}$$

$$\nabla_X \vec{u} = \hat{e}_1\left[\frac{\partial(u_1\hat{e}_1)}{\partial X_1} + \frac{\partial(u_2\hat{e}_2)}{\partial X_1} + \frac{\partial(u_3\hat{e}_3)}{\partial X_1}\right] + \hat{e}_2\left[\frac{\partial(u_1\hat{e}_1)}{\partial X_2} + \frac{\partial(u_2\hat{e}_2)}{\partial X_2} + \frac{\partial(u_3\hat{e}_3)}{\partial X_2}\right] + \hat{e}_3\left[\frac{\partial(u_1\hat{e}_1)}{\partial X_3} + \frac{\partial(u_2\hat{e}_2)}{\partial X_3} + \frac{\partial(u_3\hat{e}_3)}{\partial X_3}\right] = \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{\partial u_2}{\partial X_1} & \frac{\partial u_3}{\partial X_1} \\ \frac{\partial u_1}{\partial X_2} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_3}{\partial X_2} \\ \frac{\partial u_1}{\partial X_3} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_3}{\partial X_3} \end{bmatrix}$$

Dyad (2nd-order tensor)

$$A = \sum_i \sum_j a_i b_j\, \hat{e}_i \hat{e}_j$$

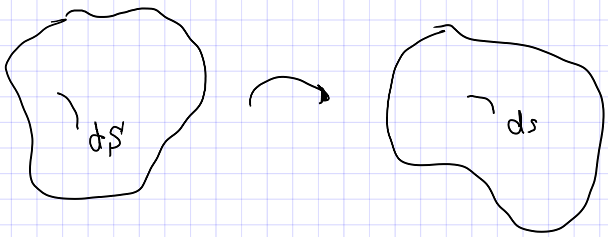
$$\nabla_X(\cdot) = \begin{bmatrix} \frac{\partial(\cdot)}{\partial X_1} \\ \frac{\partial(\cdot)}{\partial X_2} \\ \frac{\partial(\cdot)}{\partial X_3} \end{bmatrix}$$

$$\vec{x} = \vec{X} + \vec{u}(\vec{X})$$

$$\frac{\partial x_i}{\partial X_j} = \frac{\partial X_i}{\partial X_j} + \frac{\partial u_i(\vec{X})}{\partial X_j}$$

$$\frac{\partial x_i}{\partial X_j} = \delta_{ij} + \frac{\partial u_i}{\partial X_j} = F_{ij}$$

$$\boxed{F_{ij} = \frac{\partial x_i}{\partial X_j}} = \frac{\partial x_i(X_1, X_2, X_3)}{\partial X_j}$$



$$(ds)^2 = (|d\vec{x}|)^2 = \left(\sqrt{dx_1^2 + dx_2^2 + dx_3^2}\right)^2 = dx_1^2 + dx_2^2 + dx_3^2 = [dx_1 \ dx_2 \ dx_3]\begin{bmatrix} dx_1 \\ dx_2 \\ dx_3 \end{bmatrix} = d\vec{x}^T d\vec{x}$$

$$ds^2 = d\vec{x}^T d\vec{x}$$

$$dS^2 = d\vec{X}^T d\vec{X}$$

$$\begin{aligned} (ds)^2 - (dS)^2 &= d\vec{x}^T d\vec{x} - d\vec{X}^T d\vec{X} \\ &= (F d\vec{X})^T (F d\vec{X}) - d\vec{X}^T d\vec{X} \\ &= d\vec{X}^T F^T F \, d\vec{X} - d\vec{X}^T I \, d\vec{X} \\ &= d\vec{X}^T \underbrace{\boxed{(F^T F - I)}}_{\equiv 2E} d\vec{X} \end{aligned}$$

$E = \frac{1}{2}(F^T F - I) \rightarrow$ Lagrangian strain
Green - St. Venant strain

$$F = I + (\nabla_X u)^T$$

$$F^T = I + \nabla_X u$$

$$\begin{aligned} E &= \frac{1}{2}\left[\nabla_X u + (\nabla_X u)^T + (\nabla_X u)^T (\nabla_X u)\right] \\ &= \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \cancelto{0}{\frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}}\right] \end{aligned}$$

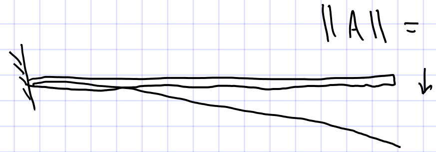
$$\varepsilon = \frac{1}{2}\left[\nabla u + (\nabla u)^T\right] = \frac{1}{2}\left[\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right]$$

Linear strain
Cauchy strain

"small"

$$\|\nabla_X u\| \ll 1$$

$$\|A\| = \sqrt{A_{ij} A_{ij}}$$



$$F^{-1} d\vec{x} = F^{-1} F \, d\vec{X}$$

$$F^{-1} d\vec{x} = d\vec{X}$$

$$\begin{aligned} (ds)^2 - (dS)^2 &= d\vec{x}^T d\vec{x} - d\vec{X}^T d\vec{X} \\ &= d\vec{x}^T d\vec{x} - (F^{-1} d\vec{x})^T (F^{-1} d\vec{x}) \\ &= d\vec{x}^T d\vec{x} - d\vec{x}^T F^{-T} F^{-1} d\vec{x} \\ &= d\vec{x}^T \underbrace{\boxed{(I - F^{-T} F^{-1})}}_{\equiv 2e} d\vec{x} \end{aligned}$$

$$e = \frac{1}{2}\left[I - F^{-T} F^{-1}\right]$$

Eulerian, Almansi

$$e = \frac{1}{2}\left[I - F^{-T} F^{-1} \right]$$

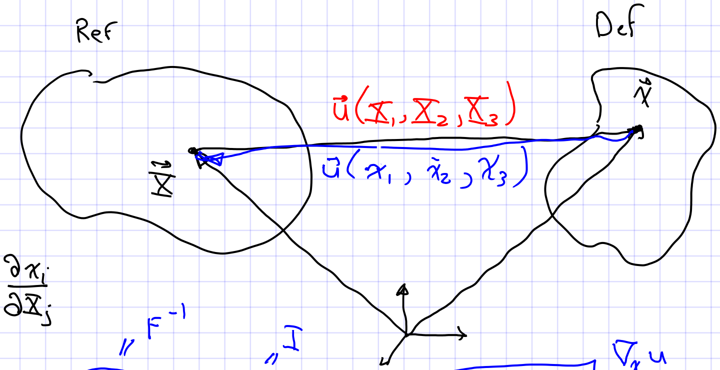
$$= \frac{1}{2}\left[\nabla_{\vec{x}} \vec{u} + (\nabla_{\vec{x}} \vec{u})^T + \underbrace{(\nabla_{\vec{x}} \vec{u})^T (\nabla_{\vec{x}} \vec{u})}_{0} \right]$$

$$= \frac{1}{2}\left[\nabla_{\vec{x}} \vec{u} + (\nabla_{\vec{x}} \vec{u})^T \right]$$

$$= \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right]$$

$$\vec{x} = \vec{X} + \vec{u}(\vec{X})$$

$$\vec{x} = \vec{X}$$



$$F = \frac{\partial x_i}{\partial X_j}$$

$$\overbrace{\boxed{\frac{\partial \vec{X}}{\partial \vec{x}}}}^{F^{-1}} = \overbrace{\boxed{\frac{\partial \vec{x}}{\partial \vec{x}}}}^{I} + \overbrace{\boxed{\frac{\partial \vec{u}(x_1, x_2, x_3)}{\partial \vec{x}}}}^{\nabla_x u}$$

$$F^{-1} = I + \nabla_{\vec{x}} \vec{u}$$