$$\frac{D}{Dt}(\rho_0) = \frac{D}{Dt}(\rho J)$$

$$0 = \frac{D\rho}{Dt}J + \rho\frac{DJ}{Dt}$$

$$= \frac{D\rho}{Dt}J + \rho J\,\nabla\cdot\vec{v}$$

$$= \frac{\partial\rho}{\partial t} + \nabla\rho\cdot\vec{v} + \rho\,\nabla\cdot\vec{v}$$

$$\boxed{0 = \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v})}$$

Conservation of mass

$$F_{ij} = \frac{\partial x_i}{\partial X_j}$$

$$\frac{D}{Dt}(J) = \frac{D}{Dt}(\det(F))$$

$$= \underbrace{\frac{\partial}{\partial F_{ij}}(\det F_{ij})}\frac{\partial F_{ij}}{\partial t}$$

$$= \underbrace{\det(F_{ij})}(F_{ji})^{-1}\frac{\partial F_{ij}}{\partial t}$$

$$= J\frac{\partial X_i}{\partial x_j}\frac{\partial}{\partial t}\left(\frac{\partial x_i}{\partial X_j}\right)$$

$$= J\frac{\partial X_i}{\partial x_j}\frac{\partial}{\partial X_j}\left(\cancel{\frac{\partial x_i}{\partial t}}\right) v_i$$

$$= J\frac{\partial X_i}{\partial x_j}\frac{\partial v_i}{\partial X_j}$$

$$= J\frac{\partial X_i}{\partial x_j}\frac{\partial v_i}{\partial x_k}\frac{\partial x_k}{\partial X_j}$$

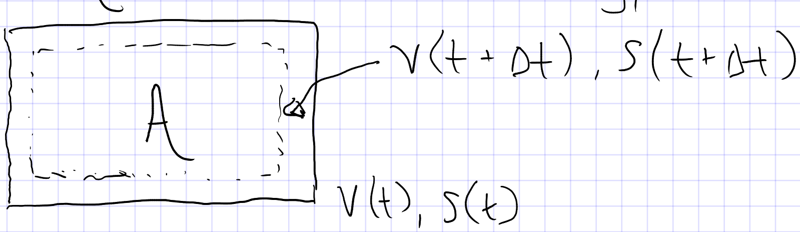
$$= J\frac{\partial X_i}{\partial x_j}\frac{\partial x_k}{\partial X_j}\frac{\partial v_i}{\partial x_k}$$

$$= J\,\delta_{ik}\frac{\partial v_i}{\partial x_k}$$

$$= J\frac{\partial v_i}{\partial x_i}$$

$$= J\,\nabla\cdot\vec{v}$$

$A \to$ mass, momentum, energy



"constant V"

$\frac{d}{dt}\int \rho A\, dV(t)$ time rate of change of A = instantaneous change A + flux A

Divergence Theorem

$$\int_S \vec{v}\cdot\hat{n}\,dS = \int_V \nabla\cdot\vec{v}\,dV$$

$$\frac{d}{dt}\int_{V(t)}\rho A\,dV = \int_{V_0}\frac{d}{dt}(\rho A)\,dV + \int_{S_0}\rho A\vec{v}\cdot\vec{n}\,dS$$

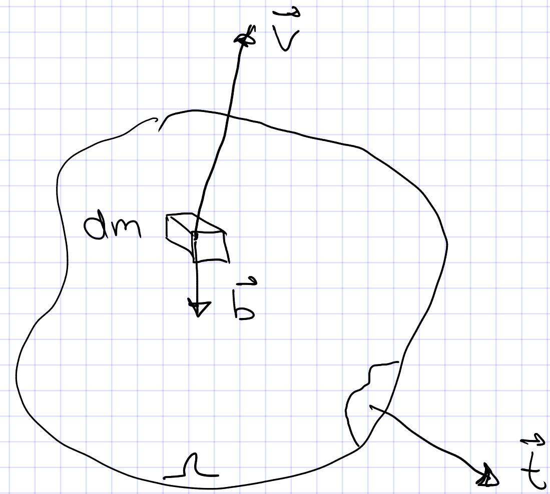
$$= \int\left[\frac{d}{dt}(\rho)A + \rho\frac{d}{dt}(A) + A\nabla\cdot(\rho\vec{v}) + \rho\vec{v}\,\nabla\cdot A\right]dV$$

$$= \int A\underbrace{\left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{v})\right)}_{=0} + \underbrace{\rho\frac{\partial A}{\partial t} + \rho\vec{v}\cdot\nabla A}_{\rho\frac{DA}{Dt}}\,dV$$

$$\boxed{\frac{d}{dt}\int_{V(t)}\rho A\,dV = \int\rho\frac{DA}{Dt}\,dV}$$

Reynold's Transport Theorem

Conservation of Linear Momentum



$$dp = \vec{v}\, dm$$

$$p = \int_\Omega \rho \vec{v}\, dv$$

$$\frac{dp}{dt} = \underbrace{\frac{d}{dt}\int_\Omega \rho \vec{v}\, dv}_{\text{R.T.T.}} = \int_\Omega \rho \vec{b}\, dv + \int_{\partial\Omega} \vec{t}\, dS$$

$$\underbrace{\int_{\partial\Omega} \vec{t}\, dS}_{\underbrace{\int_{\partial\Omega} \sigma^T \cdot \hat{n}\, dS}_{\text{D.T.}}}$$

body forces $= \int_\Omega \rho \vec{b}\, dv$

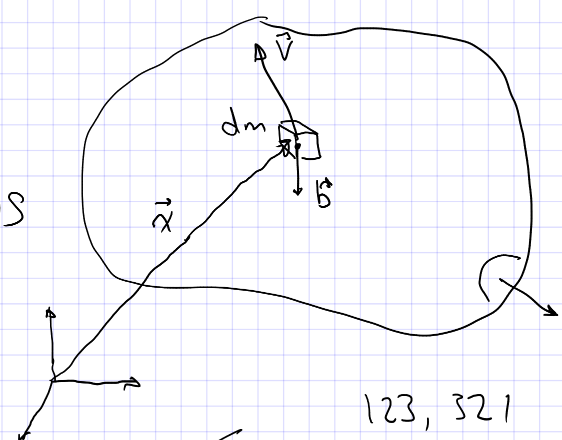
surfaces forces $= \int_{\partial\Omega} \vec{t}\, dS = \int_{\partial\Omega} \sigma^T \cdot \hat{n}\, dS$

$$\int \rho \frac{D\vec{v}}{Dt}\, dV = \int \rho \vec{b}\, dV + \int \nabla \cdot \sigma^T\, dV = \int [\rho \vec{b} + \nabla \cdot \sigma^T]\, dV$$

$$\boxed{\rho \frac{D\vec{v}}{Dt} = \rho \vec{b} + \nabla \cdot \sigma^T}$$ Cauchy momentum eqn.

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v}\right] = \rho \vec{b} + \nabla \cdot \sigma^T$$

Angular Momentum



$$\frac{d}{dt}\int_\Omega \vec{x} \times (\rho \vec{v})\, dv = \int_\Omega \vec{x} \times \rho \vec{b}\, dV + \int_{\partial\Omega} \vec{x} \times \vec{t}\, dS$$

$$\vec{a} \cdot \vec{b} = a_i b_i$$

$$\vec{a} \times \vec{b} = \varepsilon_{ijk}\, a_j b_k$$

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{123, 321 even permutations} \\ -1 & \text{132, 213 odd permutations} \\ 0 & \text{113 repeated} \end{cases}$$

permutation tensor
"Levi-Cevita" tensor

$$\underbrace{\frac{d}{dt}\int \varepsilon_{rmn}\, x_m\, v_n\, \rho\, dV}_{\text{R.T.T.}} = \int \varepsilon_{rmn}\, x_m\, t_n\, dS + \int \varepsilon_{rmn}\, x_m\, b_m\, dV$$

$$\int \varepsilon_{rmn}\, \rho\, \frac{D(x_m v_n)}{Dt}\, dV = \int \rho\, \cancelto{0}{\varepsilon_{rmn}\, v_n\, \cancelto{v_m}{\frac{Dx_m}{Dt}}}\, dV + \int \rho\, \varepsilon_{rmn}\, x_m\, \frac{Dv_n}{Dt}\, dV$$

$$\int \varepsilon_{rmn}\, x_m\, t_n\, dS = \int \varepsilon_{rmn}\, x_m\, \sigma_{jn}\, n_j\, dS \overset{\text{D.T.}}{=} \int \varepsilon_{rmn} \frac{\partial}{\partial x_j}(x_m\, \sigma_{jn})\, dV$$

$$= \int \varepsilon_{rmn}\, \sigma_{jn} \frac{\partial}{\partial x_j} x_m + \varepsilon_{rmn}\, x_m \frac{\partial \sigma_{jn}}{\partial x_j}\, dV$$

$$\int \rho\, \varepsilon_{rmn}\, x_m \frac{Dv_n}{Dt}\, dV = \int \varepsilon_{rmn}\, \sigma_{jn} \frac{\partial x_m}{\partial x_j} + \varepsilon_{rmn}\, x_m \frac{\partial \sigma_{jn}}{\partial x_i}\, dV + \int \varepsilon_{rnm}\, x_m\, \rho\, b_n\, dV$$

$$\int \varepsilon_{rmm}\, x_m \underbrace{\left[\rho \frac{Dv_n}{Dt} - \frac{\partial \sigma_{jn}}{\partial x_j} - \rho\, b_n\right]}_{=0} dV = \int \varepsilon_{rnm}\, \sigma_{jn} \underbrace{\frac{\partial x_m}{\partial x_j}}_{\delta_{mj}}\, dV$$

$$0 = \int \varepsilon_{rmn}\, \sigma_{mn}\, dV = \varepsilon_{rmn}\, \sigma_{mn}$$

$r = 1 \rightarrow 0 = \sigma_{23} - \sigma_{32}$

$r = 2 \rightarrow 0 = \sigma_{31} - \sigma_{13}$

$r = 3 \rightarrow 0 = \sigma_{21} - \sigma_{12}$

$\boxed{\sigma^T = \sigma}$ or $\boxed{\sigma_{jm} = \sigma_{mj}}$

σ is symm.