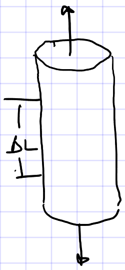


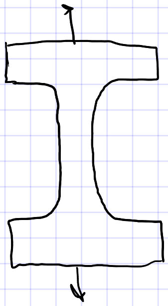

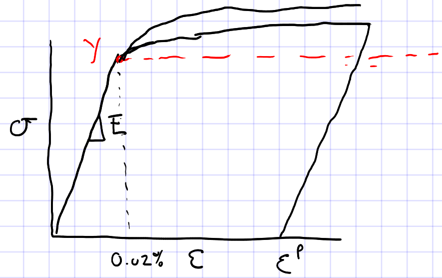


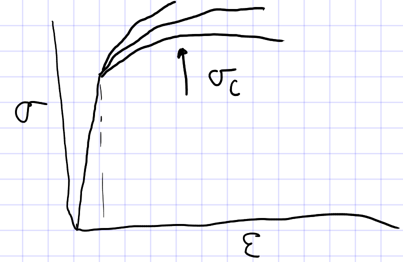


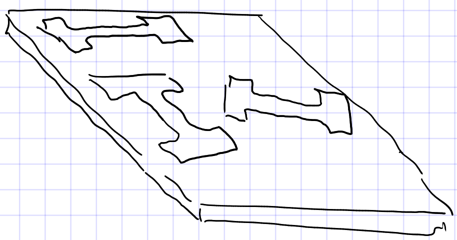

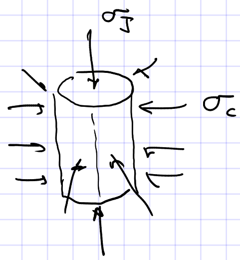


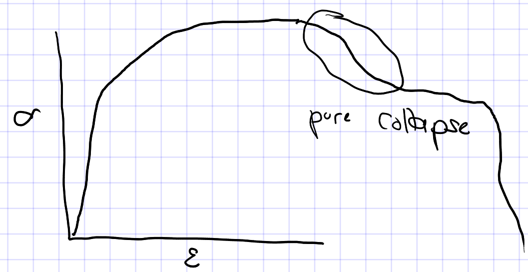


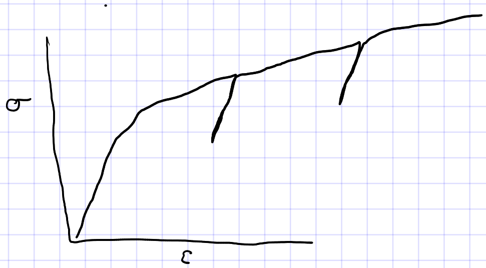


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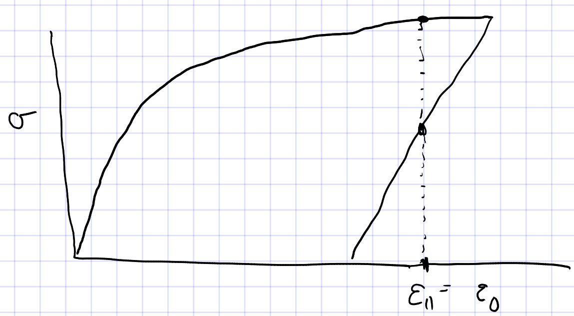
$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} \qquad W = \rho\Psi = W(\varepsilon_{ij}, T)$$




$$\varepsilon_{ij} = \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sigma = \sigma(\varepsilon_{ij}, T, \vec{\xi})$$

$\vec{\xi}$ internal state variables

---

$\vec{\xi}$ could be "physical varible"

- → structure
- → physio-chemical reactions
- → phase changes
- → densities of structural defects
- → phenomenological
  - → eg plastic strain

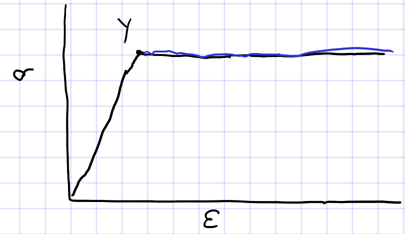
$$\bar{\bar{\varepsilon}} = \bar{\bar{\varepsilon}}^e + \bar{\bar{\varepsilon}}^p$$

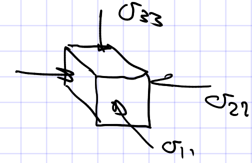
small strains $\|\nabla u\| << 1$

$$\dot{\bar{\bar{\varepsilon}}} = \dot{\bar{\bar{\varepsilon}}}^e + \dot{\bar{\bar{\varepsilon}}}^p$$

$$\dot{\varepsilon}^e = \dot{\varepsilon} - \dot{\varepsilon}^p$$

$$d\varepsilon^e = d\varepsilon - d\varepsilon^p$$


$\sigma_{11} = 10$ MPa

$\sigma_{22} = \sigma_{33} = 20$ MPa

$Y = 15$ MPa

Is the material yielding?

$$\sigma = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\sigma_{11} = E\,\varepsilon_{11}$ (elastic)

$\sigma_{11} = Y$ (plastic)

$$S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} = \begin{bmatrix} Y & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} Y/3 & 0 & 0 \\ 0 & Y/3 & 0 \\ 0 & 0 & Y/3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3}Y & 0 & 0 \\ 0 & -\frac{1}{3}Y & 0 \\ 0 & 0 & -\frac{1}{3}Y \end{bmatrix} = S_{ij}$$

$$\sigma_{eq} \equiv \sqrt{3 J_2} = \sqrt{\frac{3}{2} S_{ij} S_{ij}} = Y$$

$$\sum_{i=1}^{3}\sum_{j=1}^{3} S_{ij}S_{ij} = \sum_{j=1}^{3} S_{1j}S_{1j} + \sum_{j=1}^{3} S_{2j}S_{2j} + \sum_{j=1}^{3} S_{3j}S_{3j}$$

$$= S_{11}S_{11} + S_{12}\cancel{S_{12}} + S_{13}\cancel{S_{13}} + S_{21}\cancel{S_{21}} + S_{22}S_{22} + S_{23}\cancel{S_{23}}$$

$$+ S_{31}\cancel{S_{31}} + S_{32}\cancel{S_{32}} + S_{33}S_{33}$$

$$\frac{4}{9}Y^2 + \frac{1}{9}Y^2 + \frac{1}{9}Y^2 = \sqrt{\frac{3}{2}\cdot\frac{2}{3}Y^2} = Y$$

$\sigma_{eq} \Rightarrow \sigma_m \Rightarrow \sigma_{vm}$ von Mises stress

<u>von Mises Plasticity ( $J_2$ plasticity)</u>

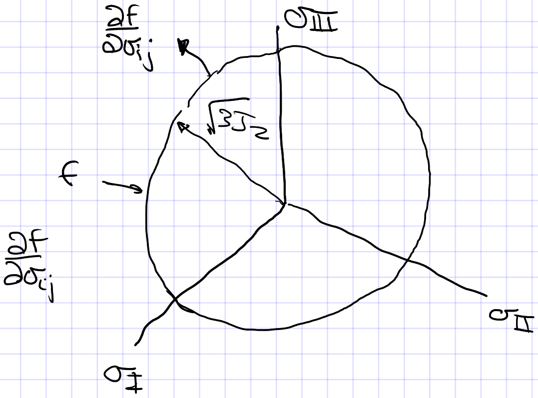
$$\sigma_{eq} = \sqrt{3J_2} = \left[\frac{1}{2}\left\{(\sigma_{11}-\sigma_{22})^2 + (\sigma_{33}-\sigma_{11})^2 + (\sigma_{22}-\sigma_{33})^2\right\} + 3\sigma_{12}^2 + 3\sigma_{13}^2 + 3\sigma_{23}^2\right]^{1/2}$$

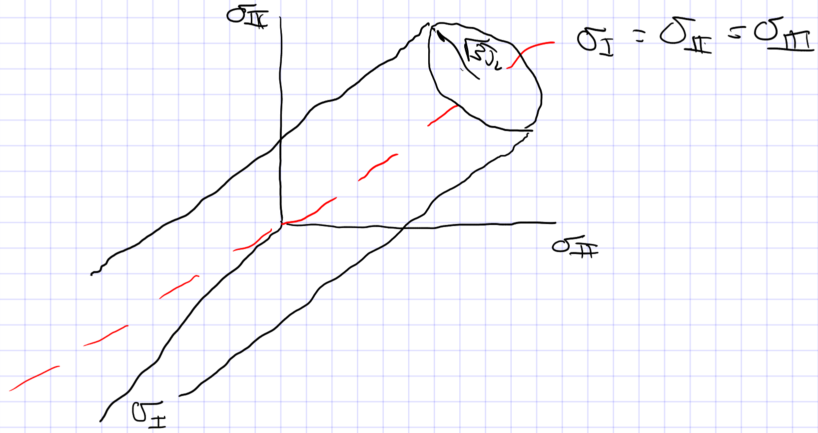
$\sigma_{eq} = 10$ MPa $\quad Y = 15$ MPa $\quad \therefore$ not yielding

$$f(\sigma_{ij}) = \sqrt{3J_2} - Y = 0$$

$f(\sigma_{ij}) < 0 \Rightarrow$ elastic

$f(\sigma_{ij}) = 0 \Rightarrow$ plastic

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$$\varepsilon_{11}^e = \frac{\sigma_{11}}{E} - \frac{\nu}{E}\left(\cancelto{0}{\sigma_{22}} + \cancelto{0}{\sigma_{33}}\right) = \frac{\sigma_{11}}{E} = \frac{Y}{E}$$

$$\varepsilon_{22}^e = \frac{-\nu}{E}\sigma_{11} = \frac{-\nu Y}{E}$$

$$\varepsilon_{33}^e = \frac{-\nu}{E}\sigma_{11} = \frac{-\nu Y}{E}$$

$$\varepsilon = \varepsilon^e + \varepsilon^p$$

## Flow rule

$$\dot{\varepsilon}^p = \dot{\lambda}\frac{\partial f}{\partial \sigma_{ij}}$$

$$d\varepsilon^p = d\lambda \frac{\partial f}{\partial \sigma_{ij}}$$

$$f = \sqrt{3J_2} - Y = 0$$
$$\sqrt{3J_2} = Y$$
$$3J_2 = Y^2$$
$$f = J_2 - \frac{Y^2}{3}$$

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial}{\partial \sigma_{ij}}\left(J_2 - \frac{Y^2}{3}\right)$$
$$= \frac{\partial}{\partial \sigma_{ij}}(J_2) = \frac{\partial}{\partial \sigma_{ij}}\left(\frac{1}{2}S_{kl}S_{kl}\right) \leftarrow$$
$$= \frac{1}{2}\left(\frac{\partial S_{kl}}{\partial \sigma_{ij}}S_{kl} + S_{kl}\frac{\partial S_{kl}}{\partial \sigma_{ij}}\right) \leftarrow$$
$$= S_{kl}\frac{\partial S_{kl}}{\partial \sigma_{ij}}$$
$$= S_{kl}\frac{\partial}{\partial \sigma_{ij}}\left(\sigma_{kl} - \frac{1}{3}\sigma_{mm}\delta_{kl}\right)$$
$$= S_{kl}\left(\frac{\partial \sigma_{kl}}{\partial \sigma_{ij}} - \frac{1}{3}\frac{\partial \sigma_{mm}}{\partial \sigma_{ij}}\delta_{kl}\right)$$
$$= S_{kl}\left(\delta_{il}\delta_{kj} - \frac{1}{3}\underbrace{\delta_{im}\delta_{jm}}_{\delta_{ij}}\delta_{kl}\right)$$
$$= S_{ki}\delta_{kj} - \frac{1}{3}S_{ll}\delta_{ij}$$
$$= S_{ij} - \frac{1}{3}\cancelto{0}{S_{ll}}\delta_{ij}$$
$$= S_{ij}$$

"associated flow rule"

$$\boxed{d\varepsilon^p = d\lambda\frac{\partial f}{\partial \sigma_{ij}} = d\lambda\, S_{ij}}$$