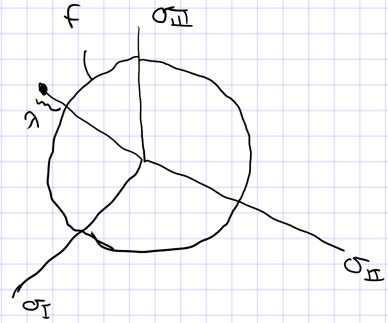
$$\dot{\varepsilon}^p = \dot{\lambda}\,\frac{\partial f}{\partial \sigma_{ij}}$$

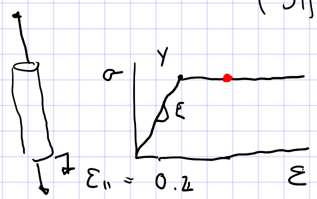
$$d\varepsilon^p = d\lambda\,\frac{\partial f}{\partial \sigma_{ij}} = \frac{\frac{\partial f}{\partial \sigma_{ij}}}{\left|\frac{\partial f}{\partial \sigma_{ij}}\right|} = d\lambda\,\frac{S_{ij}}{|S_{ij}|} = d\lambda\, Q_{ij}$$

$$|S_{ij}| = \sqrt{S_{ij} S_{ij}}$$



$$d\varepsilon = d\lambda \begin{bmatrix} \frac{2Y}{3} & 0 & 0 \\ 0 & -Y/3 & 0 \\ 0 & 0 & -\frac{Y}{3} \end{bmatrix}$$

ε_{11}^p, $-\varepsilon_{11}^p/2$, $-\varepsilon_{11}^p/2$

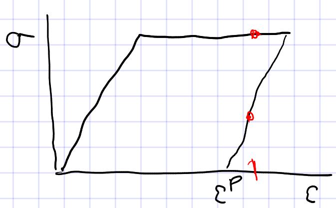


$$\varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p$$

$$\varepsilon_{11} = \frac{Y}{E} + \varepsilon_{11}^p \quad \Rightarrow \quad \varepsilon_{11}^p = \varepsilon_{11} - \frac{Y}{E}$$

$$\varepsilon_{22} = \frac{-\nu Y}{E} + \frac{1}{2}\left(\varepsilon_{11}^p\right) = \frac{-\nu}{E} Y - \frac{1}{2}\left(\varepsilon_{11} - \frac{Y}{E}\right)$$

$$\varepsilon_{33} = \frac{-\nu}{E} Y + \frac{1}{2}\left(\varepsilon_{11}^p\right) = \frac{-\nu}{E} Y - \frac{1}{2}\left(\varepsilon_{11} - \frac{Y}{E}\right)$$



"equivalent plastic strain", ε^p

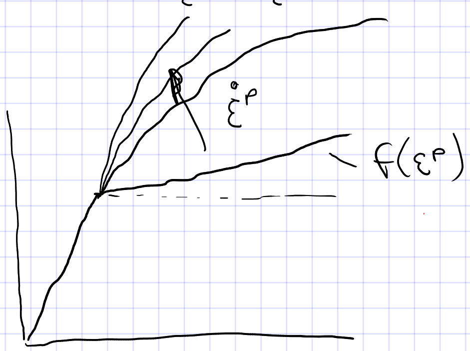
$$\dot{\varepsilon}_{ij} = \dot{\lambda}\,\frac{\frac{\partial f}{\partial \sigma_{ij}}}{\left|\frac{\partial f}{\partial \sigma_{ij}}\right|} = \dot{\lambda}\, Q_{ij}$$

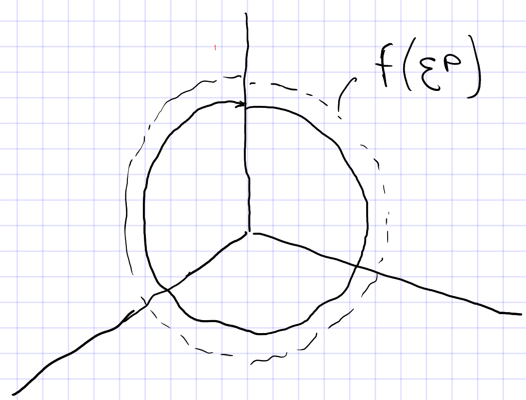
$$\dot{\varepsilon}^p = \sqrt{\frac{2}{3}\,\dot{\varepsilon}_{ij}^p\,\dot{\varepsilon}_{ij}^p}$$

$$= \sqrt{\frac{2}{3}\,\dot{\lambda}^2\,\underbrace{Q_{ij}\,Q_{ij}}_{1}}$$

$$= \sqrt{\frac{2}{3}}\,\dot{\lambda}$$

$$\varepsilon^p = \int_0^{t_p} \dot{\varepsilon}^p\, dt$$





$$e_{ij} = \varepsilon_{ij} - \frac{1}{3}\varepsilon_{kk}\delta_{ij}$$

$$S_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

$$\sigma_{ij} = 2\mu e_{ij} + K\frac{1}{3}\varepsilon_{kk}\delta_{ij}$$

$$S_{ij} = 2\mu e_{ij}$$

$$\dot{e}^e_{ij} = \frac{\dot{S}_{ij}}{2\mu}$$

$$\dot{e}^p_{ij} = \dot{\lambda} Q_{ij}$$

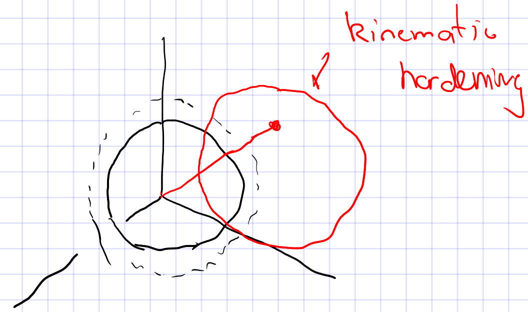
$$\dot{e}_{ij} = \dot{e}^e_{ij} + \dot{e}^p_{ij}$$

$$\dot{e}_{ij} - \dot{e}^e_{ij} - \dot{e}^p_{ij} = 0$$

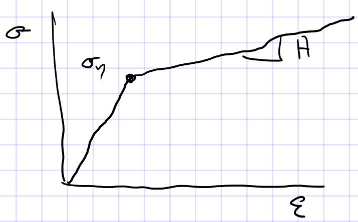
$$\dot{e}_{ij}Q_{ij} - \dot{e}^e_{ij}Q_{ij} - \dot{e}^p_{ij}Q_{ij} = 0$$

$$\dot{e}_{ij}Q_{ij} - \overbrace{\frac{\dot{S}_{ij}}{2\mu}Q_{ij}}^{\dot{S}} - \dot{\lambda}\underbrace{Q_{ij}Q_{ij}} = 0$$

$$\boxed{\dot{e}_{ij}Q_{ij} - \frac{\dot{S}}{2\mu} - \dot{\lambda} = 0} \quad \star$$



Ex Isotropic Hardening



$$f(J_2) = \sqrt{3J_2} - Y(\varepsilon^p) = 0$$

$$= \sqrt{3J_2} - \sigma_y - H\varepsilon^p$$

$f < 0$ "elastic"

$f = 0$ "plastic"

Kuhn-Tucker Constraint Equations

$$f = 0, \quad \dot{\lambda} \geq 0, \quad \dot{\lambda} f = 0$$

$$\boxed{\dot{f} = 0}$$

$$s = |S_{ij}| = \sqrt{S_{ij}S_{ij}}$$

$$\sqrt{3J_2} = \sqrt{\frac{3}{2}S_{ij}S_{ij}} = \sqrt{\frac{3}{2}}\, s$$

$$\dot{\varepsilon}^p = \sqrt{\frac{2}{3}}\,\dot{\lambda}$$

$$f(J_2) = \sqrt{3J_2} - \sigma_y - H\varepsilon^p$$

$$= \sqrt{\frac{3}{2}}\, s - \sigma_y - H\varepsilon^p$$

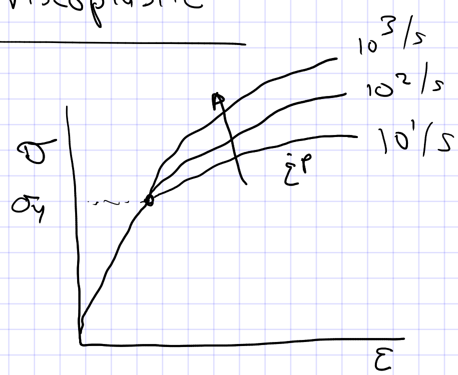
$$\dot{f} = \sqrt{\frac{3}{2}}\,\dot{s} - H\dot{\varepsilon}^p$$

$$= \sqrt{\frac{3}{2}}\,\dot{s} - H\sqrt{\frac{2}{3}}\,\dot{\lambda} = 0 \quad \Rightarrow \quad \boxed{\dot{s} = \frac{2}{3}H\dot{\lambda}} \rightarrow \star$$

$$\dot{\lambda} = \dot{e}_{ij}Q_{ij}\left(\frac{H}{3\mu} + 1\right)^{-1}$$

$$\dot{\sigma}_{ij} = \begin{cases} \left(K + \dfrac{2\mu^3}{H+3\mu}\right)\dot{\varepsilon}_{kk}\,\delta_{ij} + \left(2\mu - \dfrac{6\mu^2}{H+3\mu}\right)\dot{\varepsilon}_{ij} & f = 0 \\[2ex] K\,\dot{\varepsilon}_{kk}\,\delta_{ij} + 2\mu\,\dot{\varepsilon}_{ij} & f < 0 \end{cases}$$

Viscoplastic



$$f = \sqrt{3J_2} - \sigma_y\left(1 + \beta\,\dot{\varepsilon}^p\right)^{N} \qquad\qquad \ddot{\varepsilon}^p = \sqrt{\tfrac{2}{3}}\,\ddot{\lambda}$$

$$\dot{f} = 0$$

$$= \sqrt{\tfrac{3}{2}}\,\dot{S} - N\sigma_y\,\beta\,\ddot{\varepsilon}^p\left(1 + \beta\,\dot{\varepsilon}^p\right)^{N-1}$$

$$= \sqrt{\tfrac{3}{2}}\,\dot{S} - \sqrt{\tfrac{2}{3}}\,N\sigma_y\,\beta\,\ddot{\lambda}\left(1 + \beta\sqrt{\tfrac{2}{3}}\,\dot{\lambda}\right)^{N-1}$$

$$\dot{S} = \tfrac{2}{3}\,N\sigma_y\,\beta\,\ddot{\lambda}\left(1 + \beta\sqrt{\tfrac{2}{3}}\,\dot{\lambda}\right)^{N-1} \Rightarrow \star$$

$$\boxed{0 = \dot{e}_{ij}\,Q_{ij} - \frac{1}{3\mu}\,N\sigma_y\,\ddot{\lambda}\left(1 + \beta\sqrt{\tfrac{2}{3}}\,\dot{\lambda}\right)^{N-1} - \dot{\lambda}}$$