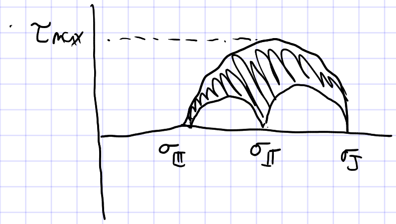
$$f(J_2) = \sqrt{3J_2} - Y(\varepsilon^p, \dot{\varepsilon}^p)$$



Other yield surfaces

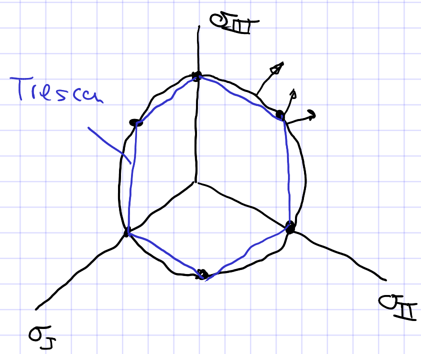
Maximum shear stress (Tresca Criterion)

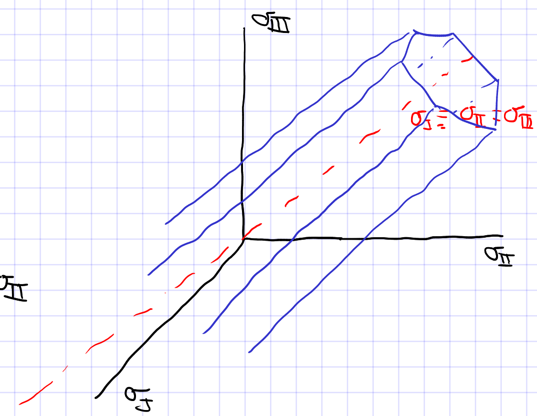
$$f = \frac{1}{2}\max\{|\sigma_I - \sigma_{II}|, |\sigma_{II} - \sigma_{III}|, |\sigma_I - \sigma_{III}|\}$$

$$= 4J_2^3 - 27J_3^2 - 9J_2^2 + Y - 6J_2Y^4 - Y^6 = 0$$

$$J_3 = \det(s_{ij}) = \sigma_I^d\,\sigma_{II}^d\,\sigma_{III}^d$$

$$s_{ij} = \begin{bmatrix} -\frac{2}{3}Y & 0 & 0 \\ 0 & +\frac{1}{3}Y & 0 \\ 0 & 0 & +\frac{1}{3}Y \end{bmatrix}$$





$J_3 = \frac{2}{27}Y^3$ tension (+)

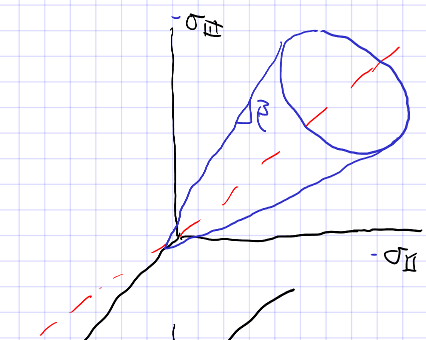
$J_3 = -\frac{2}{27}Y^3$ compression

Drucker - Prager

$$f(p, J_2) = \sqrt{3J_2} - \beta p - Y(\varepsilon^p, \dot{\varepsilon}^p, T) = 0$$

$$p = -\frac{1}{3}\sigma_{kk}$$

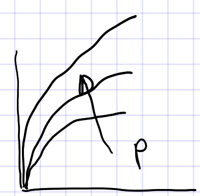
$$\frac{\partial f}{\partial \sigma_{ij}} \Rightarrow \beta\frac{\partial p}{\partial \sigma_{ij}} = \frac{\partial}{\partial \sigma_{ij}}\left(-\frac{1}{3}\sigma_{kk}\right)\beta = -\frac{1}{3}\delta_{ik}\delta_{jk}\beta = -\frac{1}{3}\delta_{ij}\beta$$



Mohr - Coulomb

$$\sigma_I - \sigma_{III} + (\sigma_I + \sigma_{III})\sin(\phi) = Y\cos(\phi)$$

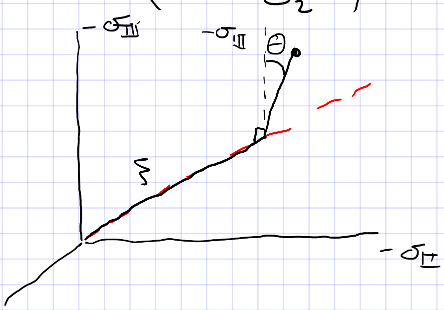
angle-of-internal friction



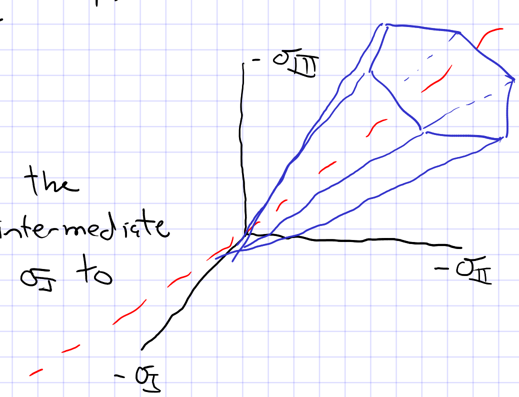
$$\frac{1}{3}I_1\sin(\phi) + \sqrt{J_2}\left\{\cos\theta - \frac{1}{\sqrt{3}}\sin\theta\sin\phi\right\} = \frac{Y}{2}\cos(\phi)$$

$\theta \rightarrow$ Lode angle

$$\theta = \frac{1}{3}\sin^{-1}\left(\frac{3\sqrt{3}}{2}\,\frac{J_3}{J_2^{3/2}}\right)$$



θ is controlled by the relationship of the intermediate priciple stress, i.e. σ_{II} to σ_{I}, σ_{III}



when $\sigma_{II} = \sigma_{III} \Rightarrow \theta = 60^\circ$

$\sigma_{II} = \sigma_{I} \Rightarrow \theta = 0^\circ$

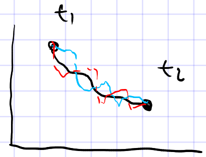
Kayenta

$$\rho \frac{Dv}{Dt} = \nabla \cdot \sigma + \rho b$$

$$\sigma = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$$

$$\rho \left(\frac{\partial v}{\partial t} + v \cdot \nabla v \right) = \nabla \cdot \sigma + \rho b$$

"Calculus of Variations"



$$I = \int_{t_1}^{t_2} f(t, y(t), y'(t))\, dt$$

$$\delta J = \int_{t_1}^{t_2} \delta f \, dt$$

$$\delta(\cdot) = \left[\frac{\partial}{\partial \varepsilon} (\cdot)^* \right]_{\varepsilon = 0}$$

where e.g.

$$q^* = q + \epsilon \delta q$$

Ex

$$\delta T(\vec{v}, t) = \left[\frac{\partial}{\partial \varepsilon} T(\vec{v} + \varepsilon \delta \vec{v}) \right]_{\varepsilon = 0}$$

$$\vec{u} = \vec{v} + \varepsilon \delta \vec{v}$$

$$= \left[\frac{\partial T}{\partial \vec{u}} \frac{\partial \vec{u}}{\partial \varepsilon} \right]_{\epsilon = 0}$$

$$\frac{\partial \vec{u}}{\partial \epsilon} = \delta \vec{v}$$

$$\vec{u}\big|_{\epsilon = 0} = \vec{v}$$

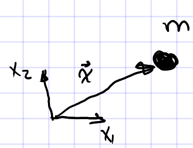
$$= \frac{\partial T}{\partial \vec{v}} \delta \vec{v}$$

Hamilton's Principle

$$\int_{t_1}^{t_2} \delta (T - U)\, dt = 0$$

$T \rightarrow$ kinetic energy

$U \rightarrow$ potential energy



$$m \ddot{x}_2 = -mg$$

$$\ddot{x}_2 = -g$$

$$T = \frac{1}{2} m \vec{v} \cdot \vec{v}$$

$$U = m g x_2$$

$$\int_{t_1}^{t_2} \delta T \, dt = \int_{t_1}^{t_2} \frac{\partial T}{\partial \vec{v}} \delta \vec{v} \, dt$$

$$= \int_{t_1}^{t_2} \frac{\partial T}{\partial \vec{v}} \delta \left(\frac{\partial \vec{x}}{\partial t} \right) dt = \int_{t_1}^{t_2} \frac{\partial T}{\partial \vec{v}} \frac{d}{dt} (\delta \vec{x})\, dt$$

$\delta \vec{x}(t_2) = \delta \vec{x}(t_1) = 0$

$$= \cancel{\left[\frac{\partial T}{\partial \vec{v}} \delta \vec{x} \right]_{t_1}^{t_2}}^{\,0} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial T}{\partial \vec{v}} \right) \delta \vec{x} \, dt$$

$$= - \int_{t_1}^{t_2} \frac{d}{dt} (m \vec{v})\, \delta \vec{x} \, dt = - \int_{t_1}^{t_2} m \vec{a} \, \delta x \, dt$$

$$\int_{t_1}^{t_2} \delta U dt = \int_{t_1}^{t_2} -mg\, \delta x_2\, dt$$

$$0 = \int_{t_1}^{t_2} \delta(T-U)\, dt = \int_{t_1}^{t_2} -m\ddot{x}_1\, \delta x_1\, dt + \int_{t_1}^{t_2} (-m\ddot{x}_2 - mg)\, \delta x_2\, dt$$

$$\int_{t_1}^{t_2} G(t)\, \delta x\, dt = 0$$

$$G(t) = 0$$

Let $\delta x = G(t)$

$$\int_{t_1}^{t_2} G(t)^2\, dt = 0$$

$$G(t) = 0$$

$$-m\ddot{x}_1 = 0$$

$$m\ddot{x}_2 = -mg$$

$$\boxed{\ddot{x}_2 = -g}$$