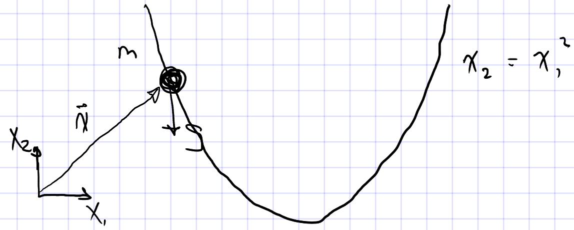
Extended Hamilton's Principle



$$0 = \int_{t_1}^{t_2} \delta(T-U) + \sum_i \delta \mathcal{C}_i \, dt$$

where $\mathcal{C}_i = \lambda_i C_i$ (no summation)

$$\mathcal{C} = \lambda(x_2 - x_1^2)$$

$$\delta C = \delta\lambda(x_2 - x_1^2) + \lambda(\delta x_2 - 2x_1 \delta x_1)$$

$$\delta\lambda: \quad x_2 - x_1^2 = 0 \xRightarrow{\frac{d}{dt}} \dot{x}_2 - 2x_1\dot{x}_1 = 0 \xRightarrow{\frac{d}{dt}} \ddot{x}_2 - 2x_1\ddot{x}_1 - 2\dot{x}_1^2 \Rightarrow \boxed{\ddot{x}_2 = 2x_1\ddot{x}_1 + 2\dot{x}_1^2}$$

$$\delta x_1: \quad -m\ddot{x}_1 - 2\lambda x_1 = 0 \quad (1)$$

$$\delta x_2: \quad -m\ddot{x}_2 - mg + \lambda = 0 \quad \leftarrow \text{multiply by } 2x_1$$

$$-2x_1 m\ddot{x}_2 - 2x_1 mg + 2\lambda x_1 = 0 \quad (2)$$

$$(1) + (2)$$

$$-2x_1 \cancel{m}\ddot{x}_2 - \cancel{m}\ddot{x}_1 - 2x_1\cancel{m}g = 0$$

$$-2x_1(2x_1\ddot{x}_1 + 2\dot{x}_1^2) - \ddot{x}_1 + 2x_1 g = 0$$

$$\boxed{\ddot{x}_1 = \frac{-2x_1 g + 4x_1\dot{x}_1^2}{4x_1^2 + 1}}$$

Integration-by-parts

$$\int_a^b w \frac{dv}{dx}\,dx = -\int v \frac{dw}{dx}\,dx + w(b)v(b) - w(a)v(a)$$

We can establish this by

$$\frac{d}{dx}(wv) = \frac{dw}{dx}v + w\frac{\partial v}{\partial x} \Rightarrow w\frac{dv}{dx} = \frac{d}{dx}(wv) - \frac{dw}{dx}v$$

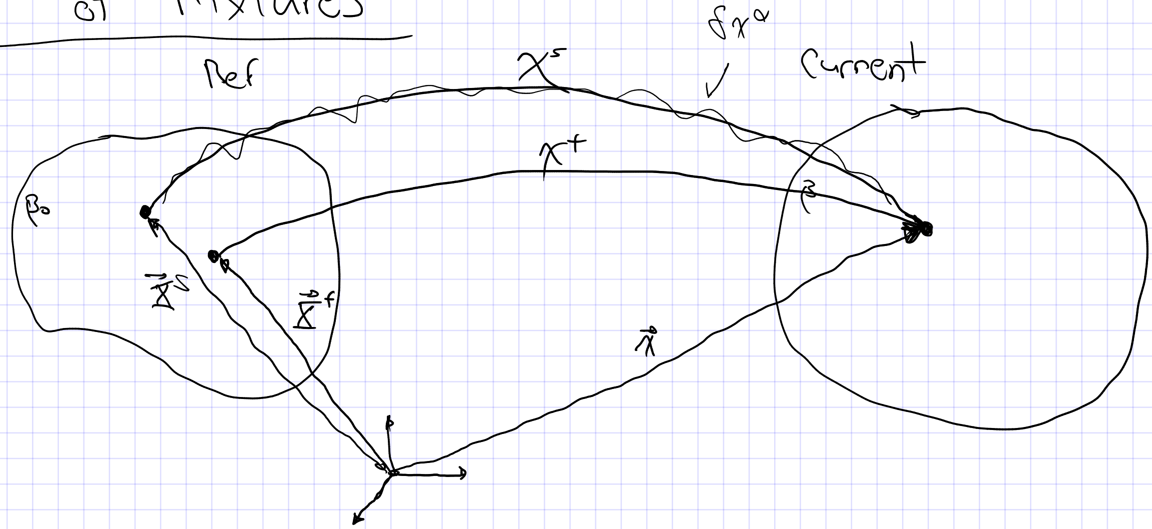
$$\int_a^b w\frac{dv}{dx} = \int_a^b \frac{d}{dx}(wv) - \frac{dw}{dx}v\,dx$$

$$= [wv]_a^b - \int v\frac{dw}{dx}\,dx$$

In higher dimensions

$$\int_\Omega u(\nabla\cdot F)\,dV = -\int_\Omega F\cdot\nabla u\,dV + \int_{\partial\Omega} uF\,dS$$

Mechanics of Mixtures



$$\vec{x} = \chi^\alpha(\vec{X}^\alpha, t) \qquad \text{for} \quad \alpha = s, f$$

$$F^\alpha = \frac{\partial \chi^\alpha}{\partial \vec{X}^\alpha} = \frac{\partial \vec{x}}{\partial \vec{X}^\alpha}$$

$$dv = J^\alpha \, dV_0^\alpha \qquad \text{where} \quad J^\alpha = \det(F^\alpha)$$

Conservation of mass

$$\int_{\beta_0} \rho_0^\alpha(\vec{X}^\alpha)\, dV_0^\alpha = \int_\beta \rho^\alpha(\vec{x}, t)\, dv \qquad \rho^\alpha = \frac{dm^\alpha}{dv}$$

subscript 0 indicates evaluation in the reference configuration

$$\int_{\beta_0} \rho_0^\alpha \, dV_0^\alpha = \int_{\beta_0} \rho^\alpha J^\alpha \, dV_0$$

$$\boxed{J^\alpha = \frac{\rho_0^\alpha}{\rho^\alpha}}$$

Now introduce

$$\rho^\alpha = \phi^\alpha(\vec{x}, t)\, \bar{\rho}^\alpha(x, t) \qquad \bar{\rho}^\alpha = \frac{dm^\alpha}{dv^\alpha}$$

Also

$$\sum_\alpha \phi^\alpha = 1 \qquad \text{"volume fraction constraint"}$$

For $\alpha = s, f$

$$\phi^s + \phi^f = 1$$

$$\boxed{J^\alpha = \frac{\phi_0^\alpha \, \bar{\rho}_0^\alpha}{\phi^\alpha \, \bar{\rho}^\alpha}}$$

Extended Hamilton's Principle

$$\int_{t_1}^{t_2} \delta(T-U) + \delta W + \sum_h \delta C_h \, dt = 0$$

$\delta W \rightarrow$ "virtual work"

Constraints:

$$C_1 = \sum_{\alpha=s,f} \int_{\beta_0} \lambda^\alpha(\mathbb{X}^\alpha)\left(J^\alpha - \frac{\phi_0^\alpha \bar{\rho}_0^\alpha}{\phi^\alpha \bar{\rho}^\alpha}\right) dV_0^\alpha$$

$$\delta C_1 = \sum_\alpha \int_{\beta_0} \lambda^\alpha \delta J^\alpha - \lambda \, \delta\left(\frac{\phi_0^\alpha \bar{\rho}_0^\alpha}{\phi^\alpha \bar{\rho}^\alpha}\right) dV_0^\alpha$$

$$\delta J^\alpha = \frac{\partial J^\alpha}{\partial F_{ij}^\alpha} \delta F_{ij}^\alpha = \frac{\partial J^\alpha}{\partial F_{ij}^\alpha} \frac{\partial \delta x_i^\alpha}{\partial \mathbb{X}_j^\alpha} = J^\alpha (F_{ij}^\alpha)^{-1} \frac{\partial \delta x_i^\alpha}{\partial \mathbb{X}_j}$$

$$\delta\left(\frac{\phi_0^\alpha \bar{\rho}_0^\alpha}{\phi^\alpha \bar{\rho}^\alpha}\right) = \delta\left(\phi_0^\alpha \bar{\rho}_0^\alpha (\phi^\alpha \bar{\rho}^\alpha)^{-1}\right) = \left(\phi_0^\alpha \bar{\rho}_0^\alpha \, \delta(\phi^\alpha \bar{\rho}^\alpha)^{-1}\right) = -\frac{J^\alpha \cancel{\phi^\alpha \bar{\rho}^\alpha}(\delta\phi^\alpha \bar{\rho}^\alpha + \phi^\alpha \delta\bar{\rho}^\alpha)}{(\phi^\alpha \bar{\rho}^\alpha)^{\cancel{2}}} = -J^\alpha\left(\frac{\delta\phi^\alpha}{\phi^\alpha} + \frac{\delta\bar{\rho}^\alpha}{\bar{\rho}^\alpha}\right)$$

$$\delta C_1 = \sum_\alpha \int_{\beta_0} \lambda^\alpha J^\alpha (F_{ij}^\alpha)^{-1} \frac{\partial \delta x_i^\alpha}{\partial \mathbb{X}_j^\alpha} + \lambda^\alpha J^\alpha\left(\frac{\delta\phi^\alpha}{\phi^\alpha} + \frac{\delta\bar{\rho}^\alpha}{\bar{\rho}^\alpha}\right) dV_0^\alpha$$

$$= \sum_\alpha \int_\beta \lambda^\alpha \frac{\partial \delta x_i^\alpha}{\partial x_j} + \lambda\left(\frac{\delta\phi^\alpha}{\phi^\alpha} + \frac{\delta\bar{\rho}^\alpha}{\bar{\rho}}\right) dv$$

$$= \sum_\alpha \int_{\partial\beta} \lambda^\alpha \hat{n}_i \, \delta x_i^\alpha \, ds - \int_\beta \frac{\partial \lambda^\alpha}{\partial x_i} \delta x_i^\alpha + \lambda^\alpha\left(\frac{\delta\phi^\alpha}{\phi^\alpha} + \frac{\delta\bar{\rho}^\alpha}{\bar{\rho}^\alpha}\right) dv$$

$$= \sum_\alpha \int_{\partial\beta} \lambda^\alpha \hat{n} \cdot \delta x_i^\alpha - \int_\beta \nabla_{\vec{x}} \lambda^\alpha \cdot \delta\vec{x}^\alpha + \lambda^\alpha\left(\frac{\delta\phi^\alpha}{\phi^\alpha} + \frac{\delta\bar{\rho}^\alpha}{\bar{\rho}^\alpha}\right) dv \quad ///$$

$$C_2 = \int_\beta p(\vec{x},t)\left(\sum_\alpha \phi^\alpha - 1\right) dv$$

$$\phi^\alpha = \phi^\alpha(\vec{x}, t)$$

$$\phi^{*\alpha} = \phi^\alpha(\vec{x} + \varepsilon\delta\vec{x}) + \varepsilon\delta\phi^\alpha$$