$$\phi^{*\alpha} = \phi^\alpha + \epsilon\,\delta\phi^\alpha \qquad\qquad \sum_\alpha \phi^\alpha(\vec{x}) - 1 = 0$$

$$\vec{x}^{*\alpha} = \chi^\alpha(X^\alpha) + \epsilon\,\delta\vec{x}^\alpha$$

$$= \Upsilon(X^\alpha, \epsilon)$$

$$X^\alpha = \Upsilon^{-1}(\vec{x}^{*\alpha}, \epsilon)$$

$$0 = \sum_\alpha \frac{d}{d\epsilon}\left[\phi^\alpha(X^\alpha) + \epsilon\,\delta\phi^\alpha\right]\Big|_{\epsilon=0}$$

$$= \sum_\alpha \frac{d}{d\epsilon}\left[\phi^\alpha\left(\Upsilon^{-1}(\vec{x}^{*\alpha}, \epsilon)\right) + \epsilon\,\delta\phi^\alpha\right]\Big|_{\epsilon=0}$$

$$= \sum_\alpha \left(\left[\frac{\partial\phi^\alpha}{\partial\Upsilon^{-1}}\boxed{\frac{\partial\Upsilon^{-1}}{\partial\epsilon}}\right]\Big|_{\epsilon=0} + \delta\phi^\alpha\right)$$

$$\overset{\vec{0}}{\cancel{dX^\alpha}} = \frac{\partial\Upsilon^{-1}}{\partial\vec{x}^{*\alpha}}\,d\vec{x}^{*\alpha} + \frac{\partial\Upsilon^{-1}}{\partial\epsilon}\,d\epsilon$$

$$\frac{\partial\Upsilon^{-1}}{\partial\epsilon} = -\frac{\partial\Upsilon^{-1}}{\partial\vec{x}^{*\alpha}}\,\frac{\partial\vec{x}^{*\alpha}}{\partial\epsilon}\Big|_{X^\alpha}$$

$$= \boxed{-\frac{\partial\Upsilon^{-1}}{\partial\vec{x}^{*\alpha}}\,\delta\vec{x}^\alpha}$$

$$= \sum_\alpha\left(\left[-\frac{\partial\phi^\alpha}{\partial\Upsilon^{-1}}\,\frac{\partial\Upsilon^{-1}}{\partial\vec{x}^{*\alpha}}\cdot\delta\vec{x}^\alpha\right]\Big|_{\epsilon=0} + \delta\phi^\alpha\right)$$

$$= \sum_\alpha\left(\left[-\frac{\partial\phi^\alpha}{\partial\vec{x}^{*\alpha}}\,\delta\vec{x}^\alpha\right]\Big|_{\epsilon=0} + \delta\phi^\alpha\right)$$

$$0 = \sum_\alpha\left(-\frac{\partial\phi^\alpha}{\partial\vec{x}}\cdot\delta x^\alpha + \delta\phi^\alpha\right)$$

$$\delta C_2 = \sum_\alpha\int_{V_\beta} p(\vec{x},t)\left(\underline{\nabla_x\phi^\alpha}\cdot\delta\vec{x}^\alpha - \delta\phi^\alpha\right)dv$$

$$\int_{t_1}^{t_2}\delta T\,dt = \int_{t_1}^{t_2}\int_{\beta}\underline{\rho^s\vec{a}^s}\cdot\delta\vec{x}^s + \underline{\rho^f\vec{a}^f}\cdot\delta\vec{x}^f\,dv \qquad\qquad \vec{a}^\alpha = \frac{\partial^2\chi^\alpha(X^\alpha,t)}{\partial t^2}$$

$$U = \int_{\beta_0}\rho_0^s\,e^s(F^s,\bar{\rho}^s)\,dV_0^s + \int_{\beta_0}\rho_0^f\,e^f(\bar{\rho}^f)\,dV_0^f$$

$$\delta U = \int_{\beta_0}\rho_0^s\left(\frac{\partial e^s}{\partial\bar{\rho}^s}\,\delta\bar{\rho}^s + \frac{\partial e^s}{\partial F^s_{ik}}\,\frac{\partial\,\delta x_i^s}{\partial X_k^s}\right)dV_0^s + \int_{\beta_0}\rho_0^f\,\frac{\partial e^f}{\partial\bar{\rho}^f}\,\delta\bar{\rho}^f\,dV_0^f$$

$$= \int_\beta \rho^s\left(\frac{\partial e^s}{\partial\bar{\rho}^s}\,\delta\bar{\rho}^s + \frac{\partial e^s}{\partial F_{ik}}\,\frac{\partial\,\delta x_i^s}{\partial x_j}\,F^s_{jk}\right) + \underline{\rho^f\,\frac{\partial e^f}{\partial\bar{\rho}^f}}\,\delta\bar{\rho}^f\,dv$$

$$J^\alpha = \frac{\rho_0^\alpha}{\rho^\alpha}$$

$$\frac{1}{J^\alpha} = \frac{\rho^\alpha}{\rho_0^\alpha}$$

$$dv = J^\alpha\,dV_0^\alpha$$

$$dv = \frac{\rho_0}{\rho}\,dV_0^\alpha$$

$$= \int_{\mathcal{B}} \quad '' \quad + \sigma'^{s}_{ji} \frac{\partial \delta x_i^s}{\partial x_j} + \quad '' \quad dv$$

$$\sigma'^{s}_{ji} = \frac{1}{J^s} P'^{s}_{ik} F^s_{jk}$$

$$P'^{s}_{ij} = \rho_0^s \frac{\partial e^s}{\partial F^s_{ij}}\Big|_{\bar{\rho}^s}$$

$$\delta U = \int_{\mathcal{B}} \rho^s \frac{\partial e^s}{\partial \bar{\rho}^s} \delta \bar{\rho}^s + \rho^f \frac{\partial e^s}{\partial \bar{\rho}^s} \delta \bar{\rho}^f - \frac{\partial \sigma'^{s}_{ji}}{\partial x_j} \delta x_i^s \, dv + \int_{\partial \mathcal{B}} \sigma'^{s}_{ji} \hat{n}_j \, \delta x_i^s \, ds$$

$$= \quad '' \quad + \quad '' \quad - \nabla \cdot (\sigma'^{s})^T \cdot \delta \vec{x}^s \, dv + \int_{\partial \mathcal{B}} (\sigma'^{s})^T \cdot \hat{n} \cdot \delta \vec{x}^s \, ds$$

$$\delta W = \int_{\mathcal{B}} (\rho^s \vec{b} + \vec{h}^s) \cdot \delta \vec{x}^s + (\rho^f \vec{b} + \vec{h}^f) \cdot \delta \vec{x}^f \, dv + \int_{\partial \mathcal{B}} \vec{t}_b^{\,s} \cdot \delta \vec{x}^s - p_b^f \hat{n} \cdot \delta \vec{x}^f \, ds$$

$$\vec{h}^s + \vec{h}^f = \vec{0}$$

Invoke the arbitrariness of $\delta \vec{x}^\alpha$, $\delta \bar{\rho}^\alpha$, $\delta \phi^\alpha$

$$\delta \vec{x}^s : \quad \rho^s \vec{a}^s = \rho^s \vec{b} + \vec{h}^s - \nabla_x \lambda^s + p \nabla_x \phi^s + \nabla_x \cdot (\sigma'^{s})^T$$

$$\delta \vec{x}^f : \quad \rho^f \vec{a}^f = \rho^f \vec{b} + \vec{h}^f - \nabla_x \lambda^f + p \nabla_x \phi^f$$

$$\left.\begin{aligned} \delta \rho^s &: \quad \lambda^s = \phi^s (\bar{\rho}^s)^2 \frac{\partial e}{\partial \bar{\rho}^s} \\ \delta \rho^f &: \quad \lambda^f = \phi^f (\bar{\rho}^f)^2 \frac{\partial e}{\partial \bar{\rho}^f} \\ \delta \phi^s &: \quad \lambda^s = p \phi^s \\ \delta \phi^f &: \quad \lambda^f = p \phi^f \end{aligned}\right\} \quad \boxed{p = (\bar{\rho}^s)^2 \frac{\partial e^s}{\partial \bar{\rho}^s} = (\bar{\rho}^f)^2 \frac{\partial e^f}{\partial \bar{\rho}^f}}$$

B.C.'s
$$\left.\begin{aligned} p_b^f \hat{n} &= \lambda^f \hat{n} \\ t_b^s &= ((\sigma'^{s})^T - \lambda^s) \hat{n} \end{aligned}\right\} \quad \boxed{\begin{aligned} p_b^f \hat{n} &= p \phi^f \hat{n} \\ t_b^s &= ((\sigma'^{s})^T - p\phi^s) \hat{n} \end{aligned}}$$

B.C.'s

$$\boxed{\begin{aligned} \rho^s \vec{a}^s &= \rho^s \vec{b} + \vec{h}^s - \phi^s \nabla_x p + \nabla_x (\sigma'^{s})^T \\ \rho \vec{a}^f &= \rho^f \vec{b} + \vec{h}^f - \phi^f \nabla_x p \end{aligned}}$$

$$\rho^s \vec{a}^s + \rho^f \vec{a}^f = \rho b - \nabla_x p + \nabla_x (\sigma'^s)^T$$

$$= \rho b + \nabla_x \cdot ((\sigma'^s)^T - p I)$$

where $\rho = \rho^s + \rho^f$

$$\vec{t}_b = \vec{t}_b^{\,s} - p_b^f \hat{n}$$

$$= ((\sigma'^s)^T - p\phi^s - p\phi^f)\, \hat{n}$$

$$= ((\sigma'^s)^T - p(\phi^s + \phi^f))\, \hat{n}$$

$$= ((\sigma'^s)^T - p)\, \hat{n}$$

$$J^s \rho^s \vec{a}^s + J^s \rho^f \vec{a}^f = J^s \rho \vec{b} + \nabla_X \cdot (P'^s - p J^s (F^s)^{-T})$$ in ref. configuration

$$\vec{T}_b = (P'^s - p J^s (F^s)^{-T})\, \hat{N}$$

$$\vec{T}_b = J^s (F^s)^{-1} \vec{t}_b$$ $\hat{N}$ is a unit vector w.r.t. reference boundary