$$\boxed{\begin{aligned} 0 &= \rho \vec{b} + \nabla_{x} \cdot \left((\sigma'^s)^T - p I \right) \\ \rho &= \rho^s + \rho^f \\ \vec{t}_b &= \left((\sigma'^s)^T - p \right) \hat{n} \end{aligned}}$$

current configuration

$$\boxed{\begin{aligned} 0 &= J^s \rho \vec{b} + \nabla_{\xi^s} \cdot \left(P'^s - p J^s (F^s)^{-T} \right) \\ \vec{T}_b &= \left(P'^s - p J^s (F^s)^{-T} \right) \hat{N} \end{aligned}}$$

reference configuration

$$P'^s = \rho_0^s \left. \frac{\partial e^s(F^s, \bar{\rho}^s)}{\partial F^s} \right|_{\bar{\rho}^s}$$

$$\bar{v}^s = \frac{1}{\bar{\rho}^s}$$

$$e^s(F^s, \bar{v}^s)$$

$$de^s = \boxed{\frac{\partial e^s}{\partial F_{ij}^s} dF_{ij}^s + \frac{\partial e^s}{\partial \bar{v}^s} d\bar{v}^s}$$

$$p = (\bar{\rho}^s)^2 \frac{\partial e^s}{\partial \bar{\rho}^s} = (\bar{\rho}^s) \frac{\partial e^s}{\partial \bar{v}^s} \frac{\partial \bar{v}^s}{\partial \bar{\rho}^s} = - \frac{\partial e^s}{\partial \bar{v}^s}$$

$$\tilde{e}(F^s, p) = e^s(F^s, \bar{v}^s) + p \bar{v}^s$$

$$d\tilde{e}^s = de^s + p\, d\bar{v}^s + \bar{v}^s dp$$

$$d\tilde{e}^s = \left(\frac{\partial e^s}{\partial F_{ij}} dF_{ij} + \overset{= -p}{\boxed{\frac{\partial e^s}{\partial \bar{v}^s}}} \cancel{d\bar{v}^s} \right) + \cancel{p\, d\bar{v}^s} + \bar{v}^s dp$$

$$d\tilde{e}^s = \frac{\partial e^s}{\partial F_{ij}^s} dF_{ij}^s + \bar{v}^s dp .$$

$$\boxed{\left. \frac{\partial \tilde{e}^s}{\partial F_{ij}} \right|_p = \left. \frac{\partial e^s}{\partial F_{ij}} \right|_{\bar{v}^s}} \quad \& \quad \left. \frac{\partial \tilde{e}^s}{\partial p} \right|_{F_{ij}^s} = \bar{v}^s \qquad \rho_0^s = \frac{1}{V_0^s}$$

$$e^s = \tilde{e}^s - p \bar{v}^s \Rightarrow \underbrace{\rho_0^s \left. \frac{\partial e^s}{\partial F_{ij}^s} \right|_p}_{P''^s_{ij}} = \underbrace{\rho_0^s \left. \frac{\partial e^s}{\partial F_{ij}^s} \right|_{\bar{v}^s}}_{P'^s_{ij}} - \rho_0^s p \left. \frac{\partial \bar{v}^s}{\partial F_{ij}} \right|_p$$

$$P''^s_{ij} = P'^s_{ij} - p \frac{1}{V_0} \left. \frac{\partial \bar{v}^s}{\partial F_{ij}} \right|_p$$

$$P^{\prime s}_{ij} = P^{\prime\prime s}_{ij} + p\,\frac{1}{V_0}\,\frac{\partial \bar{v}^s}{\partial F_{ij}}\Big|_p \qquad\qquad J^s = \frac{\bar{\rho}_0^s\,\phi_0^s}{\bar{\rho}^s\,\phi^s} = \frac{\bar{V}^s\,\phi_0^s}{\bar{V}_0\,\phi^s}$$

$$P^{\prime s}_{ij} = P^{\prime\prime s}_{ij} + p\,\frac{1}{V_0}\,\frac{\partial \bar{v}^s}{\partial J^s}\,\frac{\partial J^s}{\partial F_{ij}}\Big|_p \qquad\qquad \bar{v}^s = \bar{v}^s(J^s)$$

$$P^{\prime s}_{ij} = P^{\prime\prime s}_{ij} + p\,\frac{1}{V_0}\,\frac{\partial \bar{v}^s}{\partial J^s}\Big|_p\,J^s(F_{ji})^{-1}$$

$$0 = J^s \rho\,\vec{b} + \nabla_X \cdot \left(P^{\prime\prime s} - \underbrace{\left(1 - \frac{1}{\bar{V}_0^s}\,\frac{\partial \bar{v}^s}{\partial J^s}\right)}_{B} p\,J^s (F^s)^{-T} \right)$$

$$\boxed{\begin{aligned} 0 &= J^s \rho\,\vec{b} + \nabla_X \cdot \left(P^{\prime\prime s} - B\,p\,J^s\,(F^s)^{-T} \right) \\ \vec{T}_b &= \left(P^{\prime\prime s} - B\,p\,J^s\,(F^s)^{-T} \right)\hat{N} \end{aligned}}$$

$$0 = \rho\,\vec{b} + \nabla_x \cdot \boxed{\left((\sigma^{\prime\prime s})^T - B\,p\,I\right)}$$

$$\vec{t}_b = \left((\sigma^{\prime\prime s})^T - B\,p\,I\right)\hat{n}$$

where $B = \left(1 - \frac{1}{\bar{V}_0^s}\,\frac{\partial \bar{v}^s}{\partial J^s}\Big|_p\right)$ is Biot's coefficient

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$$J = \frac{dv}{dV_0}$$

$$\bar{J} = \frac{dv^s}{dV_0^s} = \frac{\bar{v}^s}{\bar{V}_0^s}$$

$$\phi^s = \frac{dv^s}{dv} = \frac{\bar{J}\,dV_0^s}{J\,dV_0} \qquad\qquad \phi_0^s = \frac{dV_0^s}{dV_0}$$

$$\phi^s = \frac{\bar{v}^s}{\bar{V}_0^s}\,\frac{\phi_0^s}{J} \quad \text{or} \quad \frac{\bar{v}^s}{\bar{V}_0^s} = J\,\frac{\phi^s}{\phi_0^s}$$

$$\boxed{J^s = J}$$

$$\sigma_{ij} = \phi^s \bar{\sigma}_{ij}^s + \phi^f \bar{\sigma}_{ij}^f \qquad \bar{\sigma}_{ij}^\alpha \rightarrow \text{intrinsic stress}$$

$$\phi^s \bar{\sigma}_{ij}^s = \sigma_{ij} - \phi^f \bar{\sigma}_{ij}^f \qquad \phi^f = 1 - \phi^s$$

$$\phi^s \bar{\sigma}_{ij}^s = \sigma_{ij}''^s - Bp\,\delta_{ij} - (1-\phi^s)\,\bar{\sigma}_{ij}^f \qquad \bar{\sigma}_{ij}^f = -p\,\delta_{ij} \quad (\star\star)$$

$$\sigma_{ij}''^s = \left(K - \tfrac{2}{3}G\right)\varepsilon_{kk}\,\delta_{ij} + 2G\varepsilon_{ij} \quad (\star)$$

Plug in $(\star)$ + $(\star\star)$ and evaluate at $i=j$

$$\phi^s \sigma_{ii}''^s = \left(K - \tfrac{2}{3}G\right)\varepsilon_{kk}\,\delta_{ii} + 2G\varepsilon_{kk} + (1-B)p\,\delta_{ii} - \phi^s p\,\delta_{ii}$$

use $\delta_{ii} = 3$

$$= (3K - \cancel{2G})\cancel{\varepsilon_{kk}} + \cancel{2G\varepsilon_{kk}} + 3(1-B)p - 3\phi^s p$$

$$\frac{\phi^s \sigma_{ii}''^s}{-3\phi^s} = 3K\varepsilon_{kk} + 3(1-B)p - 3\phi^s p$$

$$-\frac{1}{3}\sigma_{kk}''^s = -\frac{K}{\phi^s}\varepsilon_{kk} - \frac{1}{\phi^s}(1-B)p + p \qquad \bar{p}^s = -\frac{1}{3}\bar{\sigma}_{kk}''^s$$

$$\bar{p}^s = -\frac{K}{\phi^s}\varepsilon_{kk} - \frac{1}{\phi^s}(1-B)p + p$$

under small strains $\varepsilon_{kk} \approx (J-1)$, $\phi^s \approx \phi_0^s$, $\bar{V}_0^s \approx \bar{V}^s$

$$\bar{p}^s = -\frac{K}{\phi_0^s}(J-1) - \frac{1}{\phi_0}(1-B)p + p \qquad J = J(\bar{p}^s)$$

$$B = 1 - \frac{1}{V_0^s}\left.\frac{\partial \bar{V}^s}{\partial J}\right|_p \qquad \frac{1}{V_0^s} = \rho_0^s = \phi_0^s\bar{\rho}_0^s = \frac{\phi_0^s}{\bar{V}_0^s} \approx \frac{\phi_0^s}{\bar{V}^s}$$

$$= 1 - \frac{\phi_0^s}{\bar{V}^s}\frac{\partial \bar{V}^s}{\partial \bar{p}^s}\left.\frac{\partial \bar{p}^s}{\partial J}\right|_p \qquad \boxed{\left.\frac{\partial \bar{p}^s}{\partial J}\right|_p = -\frac{K}{\phi_0^s}} + \frac{1}{\phi_0^s \bar{V}_0^s}\cancel{\frac{\partial^2 \bar{V}^s}{\partial J^2}} \rightarrow \approx 0$$

$$\hookrightarrow -\frac{1}{K^s} = \frac{1}{\bar{V}^s}\frac{\partial \bar{V}^s}{\partial \bar{p}^s}$$

$$\boxed{\begin{aligned} B &= 1 - \frac{K}{K^s} \leftarrow \text{Biots coefficient} \\ 0 &= \rho\vec{b} + \nabla_x \cdot \left(\sigma''^s - BpJ\right) \end{aligned}}$$