Variational Formulations

Classicaly "variational formulation" refers to constructing a functional or a variational principle that is equivalent to the governing equation.

The modern use refers to a ~~formulation~~ formulation where the governing eqns are translated into an equivalent weighted-integral statement.

Weighted Integral Statement

$$u \approx u^n = \sum_{j=1}^{n} N_j u_j + \sum_{j=1}^{m} \phi_j c_j$$

$u_j \rightarrow$ "nodes", but we ~~can~~ have no "nodes".

$$u \approx u^n = \sum_{j=1}^{m} c_j \phi_j + \phi_0$$

$\leftarrow$ sole purpose is to satisfy the B.C.'s

Consider the O.D.E.

$$-\frac{d}{dx}\left[a(x)\frac{du}{dx}\right] + c(x)u = f(x) \qquad 0 < x < L$$

$$u(0) = u_0 \quad , \quad \left[a(x)\frac{du}{dx}\right]_{x=L} = Q_0$$

Let $L = 1$, $u_0 = 1$, $Q_0 = 0$

$a(x) = x$, $c(x) = 1$, $f(x) = 0$

Let's choose

$\phi_0 = 1$

$\phi_1 = x^2 - 2x$

$\phi_2 = x^3 - 3x$

$$-\frac{d}{dx}\left[x\frac{du}{dx}\right] + u = 0$$

$$\boxed{\begin{gathered} -\frac{du}{dx} - x\frac{d^2u}{dx^2} + u = 0 \\ u(0) = 1 \quad , \quad x\frac{du}{dx}\Big|_{x=L} = 0 \end{gathered}}$$

$$u \simeq u^h = c_1 (x^2 - 2x) + c_2 (x^3 - 3x) + 1$$

$$\frac{du}{dx} = c_1 (2x - 2) + c_2 (3x^2 - 3)$$

$$\frac{d^2u}{dx^2} = 2c_1 + 6x\,c_2$$

$$-c_1(2x-2) - c_2(3x^2-3) - 2x c_1 - 6c_2 x^2 + c_1(x^2 - 2x) + c_2(x^3 - 3x) + 1 = 0$$

$$x^3 : \quad c_2 = 0$$

$$x^2 : \quad -3c_2 - 6c_2 + c_1 = -9c_2 + c_1 = 0$$

$$x^1 : \quad -2c_1 - 2c_1 - 2c_1 - 3c_2 = -6c_1 - 3c_2 = 0$$

$$x^0 : \quad 2c_1 + 3c_2 + 1 = 0$$

Go back

$$\int_0^L \underbrace{\delta u}_{\omega} \underbrace{\left[-\frac{d}{dx}\left[x \frac{du}{dx} \right] + u \right]}_{R} dx = \int_0^L \delta u \left[\cancel{0} \right] dx \quad 0 \quad \Rightarrow \int_0^1 \omega R \, dx = 0$$

$$R = c_2 x^3 + (c_1 - 9c_2) x^2 + (-6c_1 - 3c_2) x + 2c_1 + 3c_2 + 1$$

choose $\delta u_1 = 1$, $\delta u_2 = x$

$$0 = \int_0^1 1 \cdot R \, dx = (1 + 2c_1 + 3c_2) + \frac{1}{2}(-6c_1 - 3c_2) + \frac{1}{3}(c_1 - 9c_2) + \frac{1}{4} c_2$$

$$0 = \int_0^1 x \cdot R \, dx = (1 + 2c_1 + 3c_2) + \frac{1}{3}(-6c_1 - 3c_2) + \frac{1}{4}(c_1 - 9c_2) + \frac{1}{5} c_2$$

$$c_1 = \frac{222}{23}, \quad c_2 = -\frac{100}{23}$$

Depending on the choice of δu_i we arrive at the different weighted residual methods. If we choose δu_i

$\delta u_i = \phi_i \Rightarrow$ Galerkin Method

$\delta u_i \neq \phi_i \Rightarrow$ Petrov-Galerkin method

$\delta u_i = \frac{d}{dx}\left(a(x) \frac{d\phi_i}{dx} \right) \Rightarrow$ least-squares method

$\delta u_i = \Delta(x - x_i) \Rightarrow$ collocation method

where $\Rightarrow$ Δ is a Dirac Delta function

$\Delta = 0 \quad x \neq x_i$

$\Delta = 1 \quad x = x_i$

Only L-S methods result in a symm. coeff matrix

$$\begin{bmatrix} 7/3 & -5/4 \\ -3/4 & -31/20 \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} \leftarrow \text{not symm,}$$

Ritz or Rayleigh-Ritz method

utilizes a quadratic functional referred to a the "weak form"

Weak Form

Step 1 Same as W-R

$$-\frac{d}{dx}\left[a(x) \frac{du}{dx} \right] = f(x) \quad \text{for } 0 < x < L$$

subject to

$u(0) = u_0$ "Essential" "Dirichlet"

$a \frac{du}{dx}\Big|_{x=L} = Q_L$ "Natural" "Nueman"

$$\int_0^L \delta u \left\{ -\frac{d}{dx}\left[a(x) \frac{du}{dx} \right] - f(x) \right\} dx = 0$$

Step 2 Integrate by parts

$$0 = \int_0^L \left\{ a(x) \frac{d(\delta u)}{dx} \frac{du}{dx} - \delta u \, f(x) \right\} dx - \left[\delta u \, a(x) \frac{du}{dx} \right]_0^L$$

Step 3 Impose B.C.'s

$$0 = \int_0^L \left\{ a(x) \frac{d}{dx}(\delta u) \frac{du}{dx} - \delta u \, f(x) \right\} dx - \delta u \, Q_L$$

Weak Form $\leftrightarrow$ Variational Form

Bilinear Form

The weak form will contain 2 types of expressions, those involving δu + u and those involving only δu.

Group

$$B(\delta u, u) = \int_0^L a(x) \frac{d}{dx}(\delta u) \frac{du}{dx} \, dx$$

$$l(\delta u) = \int_0^L \delta u \, f(x) \, dx + \delta u \, Q_L$$

$f(x)$

We can write: the problem is state as, find u;

$B(\delta u, u) = l(\delta u)$ "variational problem"

The functional $B(\delta u, u)$ is said to be <u>bilinear</u>

$$B(\alpha u_1 + \beta u_2, v) = \alpha B(u_1, v) + \beta B(u_2, v)$$

$$B(u, \alpha v_1 + \beta v_2) = \alpha B(u, v_1) + \beta B(u, v_2)$$

B is symm.

$$B(u, v) = B(v, u)$$

If bilinear + symm.

$$B(\delta u, u) = \frac{1}{2} \delta B(u, u), \quad l(\delta u) = \delta l(u)$$

$$\frac{1}{2}\,\delta B(u,u) - \delta l(u) \equiv \delta J(u) = 0$$

Restate the variational problem

$$\bar{J}(u) = \frac{1}{2} B(u,u) - l(u)$$