$$B(\delta u, u) = l(\delta u)$$

$$I(u) = \frac{1}{2} B(u,u) - l(u)$$

Ritz Method

Use the "weak form". Has advantage the approximating functions (ϕ_i's) only need to satisfy the essential B.C.'s, since the natural B.C.'s are included. We seek

$$u \approx u^h = \sum_{j=1}^{n} c_j \phi_j(x)$$

$I(u^h) \leftarrow$ sub. in u^h, and integrate, $I(c_j)$

$$\frac{\partial I}{\partial c_i} = 0$$

Example

$$-\frac{\partial^2 u}{\partial x^2} + u + x^2 = 0 \quad \text{for} \quad 0 < x < 1$$

$$u(0) = 0, \quad u(1) = 0$$

$$\underbrace{\int_0^1 \left\{ \frac{d}{dx}(\delta u) \frac{du}{dx} - \delta u \, u \right\} dx}_{B(\delta u, u)} + \int_0^1 \underbrace{\delta u \, x^2 \, dx}_{l(\delta u)} = 0$$

$$I(u) = \frac{1}{2} B(u,u) - l(u) = \frac{1}{2}\int_0^1 \left[\left(\frac{\partial u}{\partial x}\right)^2 - u^2 + 2x^2 u \right] dx$$

$$u \approx u^h = c_1 x(1-x) + c_2 x^2 (1-x) + c_3 x^3 (1-x)$$

$$\frac{\partial I}{\partial c_1} = 0, \quad \frac{\partial I}{\partial c_2} = 0, \quad \frac{\partial I}{\partial c_3} = 0$$

Interpolation Functions

Again $u \approx u^h = c_j \phi_j + \phi_0$ where ϕ_0 satisfies essential B.C.'s

Otherwise the ϕ_j have to satisfy the following

1.) ϕ_i must be selected such that $B(\phi_i, \phi_j)$ is defined and non-zero

i.e. they must have proper continuity $\quad \frac{\partial^2 u}{\partial x^2} \quad u' = x$

ϕ_j must satisfy the homogenuous form of the specified B.C.'s

i.e., $u(0) = u_0$ $\quad \phi_j$ must satisfy $u(0) = 0$

2.) The set of $\{\phi_j\}$ need to be linear independent

$$\phi_1 = x(1-x), \quad \phi_2 = x^2(1-x), \quad \phi_3 \neq 2\cancel{x}^2(1-x)$$

3.) The set $\{\phi_j\}$'s must be <u>complete</u>

$\{x, x^2, x^3, x^4\} \rightarrow$ complete

$\{x, y, xy, x^2y, xy^2, x^2y^2\} \rightarrow$ complete

$\{x^3, x^5, x^{25}\} \rightarrow$ NOT complete

$\{x, x^2, xy^3\} \rightarrow$ NOT complete

Pascal's Triangle

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & x & & y & & \\ & x^2 & & xy & & y^2 & \\ x^3 & & x^2y & & y^2x & & y^3 \\ \vdots & & & & & & \vdots \end{array}$$

4.) ϕ_0 must be the lowest order function that satisfies the B.C.'s.

<u>Almost always use polynomials</u>

<u>Example</u>

$$-\frac{\partial^2 u}{\partial x^2} + u + x^2 = 0 \qquad \text{for } 0 < x < 1$$

$$u(0) = 0, \qquad u(1) = 0$$

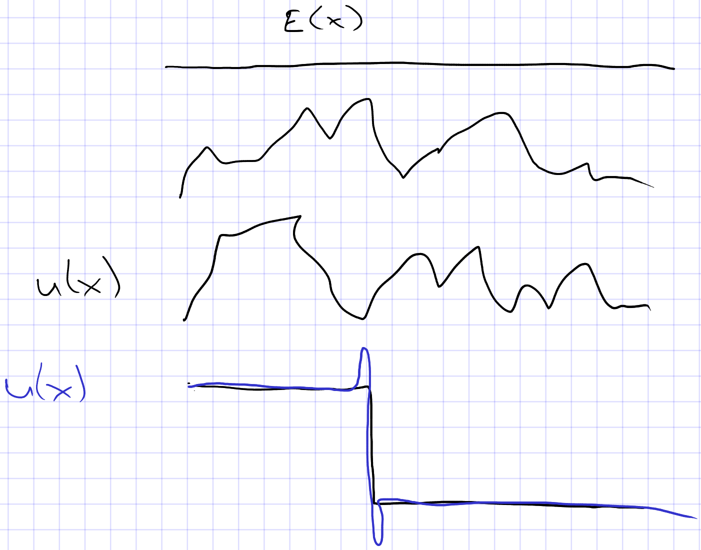
$$I(u) = \frac{1}{2}\int_0^1 \left[\left(\frac{du}{dx}\right)^2 + u^2 + 2x^2 u\right] dx$$

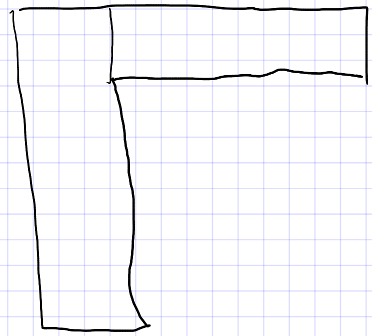
$$u \approx u^h = c_j \phi_j = c_1 + c_2 x + c_3 x^2$$

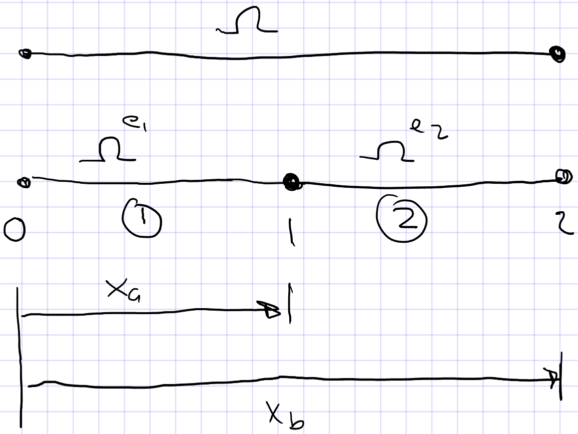
$$u(0) = 0 = c_1$$

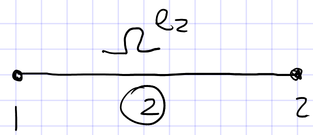
$$u(1) = 0 = c_2 + c_3 \Rightarrow c_2 = -c_3$$

$$u^h = -c_3 x + c_3 x^2 = c_3(x^2 - x)$$









$$u^h = c_1 + c_2 X$$

Evalute field @ nodes

Node 1 : $u(x_a) = c_1 + c_2 x_a \equiv u_1^{e_2}$

Node 2 : $u(x_b) = c_1 + c_2 x_b \equiv u_2^{e_2}$

$$\begin{Bmatrix} u_1^{e_2} \\ u_2^{e_2} \end{Bmatrix} = \begin{bmatrix} 1 & x_a \\ 1 & x_b \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix}$$

$$\vec{u}^{e_2} = A \quad \vec{c}$$

$$\vec{c} = A^{-1} \vec{u}$$

$$\vec{c} = \begin{bmatrix} \dfrac{u_2^{e_2} x_a - u_1^{e_2} x_b}{x_a - x_b} \\ \hline \dfrac{u_1^{e_2} - u_2^{e_2}}{x_a - x_b} \end{bmatrix} \begin{matrix} \to c_1 \\ \\ \to c_2 \end{matrix}$$

$$u^h = c_1 + c_2 X$$

$$u^h = \frac{u_2^{e_2} x_a - u_1^{e_2} x_b}{x_a - x_b} + \frac{u_1^{e_2} - u_2^{e_2}}{x_a - x_b} X$$

$$u^h = \sum_j N_j u_j = N_1 u_1^{e_2} + N_2 u_2^{e_2}$$

$$= \underbrace{\left[\frac{x - x_b}{x_a - x_b}\right]}_{N_1} u_1^{e_2} + \underbrace{\left[\frac{x_a - x}{x_a - x_b}\right]}_{N_2} u_2^{e_2}$$

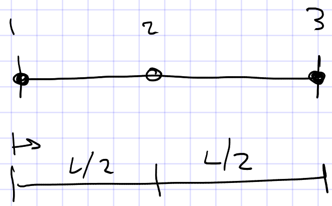
Let $x_b - x_a = L$

$$u^h = \left[1 - \frac{x}{L}\right] u_1^{e_2} + \left[\frac{x}{L}\right] u_2^{e_2}$$

$$= \underbrace{\left[1 - \frac{x}{L} ,\ \frac{x}{L}\right]}_{\vec{N}^T} \begin{Bmatrix} u_1^{e_2} \\ u_2^{e_2} \end{Bmatrix}$$

$\vec{N}^T \rightarrow$ shape function vector

Quadratic element



Let $\underline{X}^T = [1, x, x^2]$

$$u^h = N_j u_j = c_1 + c_2 x + c_3 x^2 = \underline{X}^T \vec{c}$$

$$\underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\vec{u}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & L/2 & (L/2)^2 \\ 1 & L & L^2 \end{bmatrix}}_{A} \underbrace{\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix}}_{\vec{c}}$$

$$N^T \vec{u} = \vec{\underline{X}}^T \vec{c}$$

$$N^T A \vec{c} = \vec{\underline{X}}^T \vec{c}$$

$$N^T \cancel{A A^{-1}}^{I} = \vec{\underline{X}}^T A^{-1}$$

$$N^T = \underline{X}^T A^{-1} = \underline{X}^T \begin{bmatrix} \underline{X}^T\big|_{x=x_1} \\ \underline{X}^T\big|_{x=x_2} \\ \underline{X}^T\big|_{x=x_3} \end{bmatrix}$$

$$B(\delta u, u) = \ell(\delta u)$$

$$[K]\{\vec{u}\} = \vec{F}$$