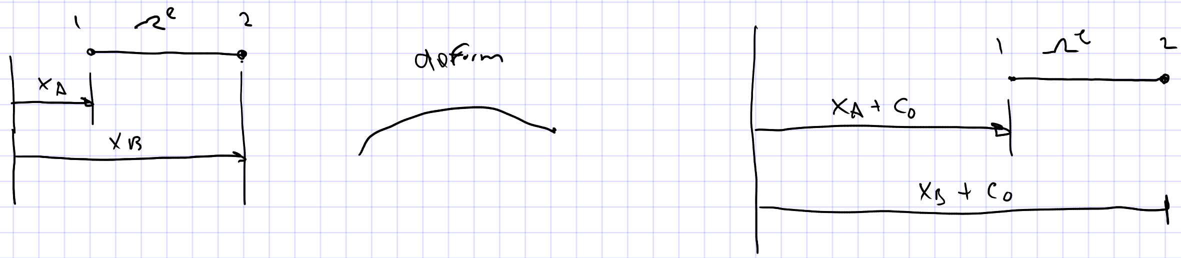
Lagrange polynomial interpolating functions

$$\|u^h - u\|_{L^2}$$

$$\Sigma (u^h_i - u_i)^2$$



$$u_1 = c_0 \quad \& \quad u_2 = c_0 \qquad u^h = c_0$$

$$u^h = \sum_j N_j u_j = N_1 u_1 + N_2 u_2 = \frac{N_1 c_0 + N_2 c_0}{c_0} = \frac{c_0}{c_0}$$

Partition-of-unity $\longrightarrow$ $N_1 + N_2 = 1$

$$N_i(x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} = \delta_{ij} \rightarrow \text{Kronecker Delta Property}$$

$$-\frac{d}{dx}\left(a \frac{du}{dx}\right) + cu - f = 0 \qquad \text{for} \quad a < x < b$$

Subject to Neumann B.C.'s

$$\left(a \frac{du}{dx}\right)\bigg|_{x=a} = Q_a \quad \& \quad \left(a \frac{du}{dx}\right)\bigg|_{x=b} = Q_b$$

Weak Form

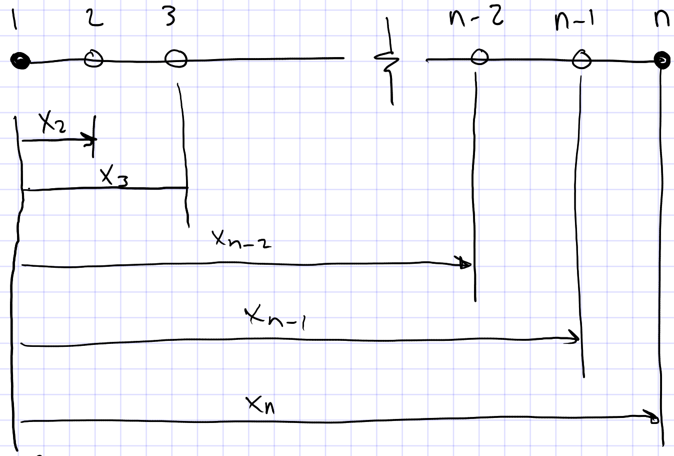
$$\int_a^b \left[a \frac{d(\delta u)}{dx} \frac{du}{dx} + c\, \delta u\, u - \delta u\, f \right] dx - \delta u(a)\, Q_a - \delta u(b)\, Q_b$$

$$B(\delta u, u) = \int_a^b \left[a \frac{d(\delta u)}{dx} \frac{du}{dx} + c\, \delta u\, u \right] dx$$

$$l(\delta u) = \int_a^b \delta u\, f\, dx + \delta u(a)\, Q_a + \delta u(b)\, Q_b$$

$$B(\delta u, u) = l(\delta u) \Rightarrow B(N_i, N_j u_j) = l(N_i)$$

$u \approx u^h = \sum_{j=1}^{n} N_j u_j$ where N_j of degree $n-1$ & u_j are the unknowns



$$0 = \sum_{i=1}^{n-1} \left\{ \int_{x_i}^{x_{i+1}} \left[a \frac{d(\delta u)}{dx} \frac{du}{dx} + \delta u \; u - \delta u \; f \right] dx - \left[\delta u(x) \; a \frac{du}{dx} \right]_{x_i}^{x_{i+1}} \right\}$$

$$= \int_{x_1}^{x_n} \left[a \frac{d(\delta u)}{dx} \frac{du}{dx} + c \; \delta u \; u - \delta u \; f \right] dx - \delta u(x_1) \left[-a \frac{du}{dx} \right]_{x_1} - \delta(x_2) \left[a \frac{du}{dx} \right]_{x_2}$$

$$- \delta u(x_2) \left[-a \frac{du}{dx} \right]_{x_2} - \cdots - \delta(x_{n-1}) \left[a \frac{du}{dx} \right]_{x_{n-1}} - \delta u(x_n) \left[a \frac{du}{dx} \right]_{x_n}$$

$$= \int_{x_1}^{x_n} \left[a \frac{d(\delta u)}{dx} \frac{du}{dx} + c \; \delta u \; u - \delta u \; f \right] dx - \delta u(x_1) Q_1 - \delta u(x_2) Q_2 - \cdots$$

$$- \delta u(x_{n-1}) Q_{n-1} - \delta u(x_n) Q_n$$

where

$$\left.\begin{aligned} Q_1 &= \left[-a \frac{du}{dx} \right]_{x_1} \\ Q_2 &= \left[\left(a \frac{du}{dx} \right)_{x_2^-} - \left(a \frac{du}{dx} \right)_{x_2^+} \right] \\ &\vdots \\ Q_{n-1} &= \left[\left(a \frac{du}{dx} \right)_{x_{n-1}^-} - \left(a \frac{du}{dx} \right)_{x_{n-1}^+} \right] \end{aligned} \right\} = 0$$

$$Q_n = \left[a \frac{du}{dx} \right]_{x_n}$$

$$0 = \int_{x_a}^{x_b} \left(a \frac{d(\delta u)}{dx} \frac{du}{dx} + c \; \delta u \; u \right) dx - \int_{x_a}^{x_b} \delta u \; f \; dx - \sum_{j=1}^{n} \delta u(x_j) \; Q_j$$

Let $u = u^h \approx N_j u_j$ + $\delta u = N_i$

For $i = 1$

$$0 = \int_{x_a}^{x_b} \left[a \frac{dN_1}{dx} \left(\frac{d}{dx} \left(N_j u_j \right) \right) + c \; N_1 \left(N_j u_j \right) \right] dx - \int_{x_a}^{x_b} N_1 \; f \; dx - \sum_{j=1}^{n} N_1(x_j) Q_j$$

$\vdots$

$$0 = \int_{x_a}^{x_b} \left[a \frac{dN_n}{dx} \frac{dN_j}{dx} u_j + c\, N_n N_j u_j \right] dx - \int_{x_a}^{x_b} N_n f \, dx - \sum_{j=1}^{n} N_n(x_j)\, Q_j$$

$\therefore$ i^{th} equation

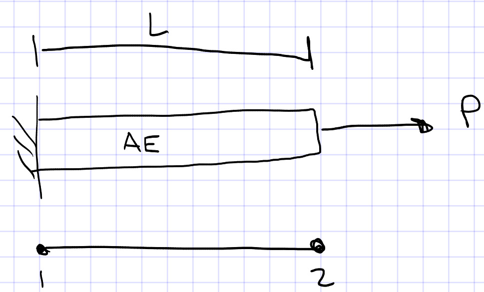
$$0 = \underbrace{\int_{x_a}^{x_b} \left[a \frac{dN_i}{dx} \frac{dN_j}{dx} + c\, N_i N_j \right] dx}\; u_j - \overbrace{\int_{x_a}^{x_b} N_i f \, dx}^{f_i} - Q_i$$

$$B(N_i, N_j)\, u_j \underbrace{- f_i - Q_i}_{F_i} = 0 \qquad N_i(x_j) = \delta_{ij}$$

$$K_{ij} u_j = F_i$$

$$\vec{u} = K^{-1} \vec{F}$$

Example



Let $(x_a, x_b) = (0, L)$

$c = 0$

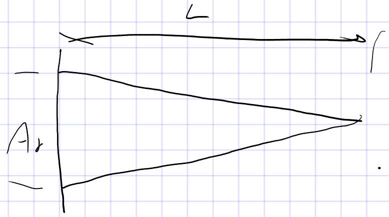
$a = AE$

$$K_{ij} = \int_0^L \left(EA \frac{dN_i}{dx} \frac{dN_j}{dx} \right) dx$$

$$\begin{bmatrix} \frac{AE}{L} & -\frac{AE}{L} \\ -\frac{AE}{L} & \frac{AE}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

$$\cancelto{1}{\frac{AE}{L}} u_1 - \cancelto{0}{\frac{AE}{L}} u_2 = 0 \quad \Rightarrow \quad u_1 = 0$$

$$-\frac{AE}{L} u_1 + \frac{AE}{L} u_2 = P$$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